Solution of ISC Specimen Questions 2021

SECTION-A

Solution 1

(i) Here, $R = \{(x, y) : x = y + 3, y > 5\}$

Now, $(7, 4) \notin R$ $[\because y > 5]$

$(9, 6) \in R$ $[\because 9 = 6 + 3]$

$(4, 1) \notin R$ $[\because y > 5]$

$(8, 5) \notin R$ $[\because y > 5]$

$\therefore$ (b) is the correct option.

(ii) Here, $A = \{1, 2, 3\}$ and $B = \{x, y\}$

$\therefore$ $n(A) = p = 3$ and $n(B) = q = 2$

Number of functions from A to $B = q^p$

$= 2^3 = 8.$

$\therefore$ (c) is the correct option.

(iii) Here, $\sin\left[\frac{\pi}{2} - \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right]$

$$= \sin\left[\frac{\pi}{2} + \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right] \qquad [\because \sin^{-1}(-x) = -\sin^{-1} x]$$

$$= \cos\left[\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right] \qquad \left[\because \sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta\right]$$

$$= \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}.$$

$\therefore$ (a) is the correct option.

(iv) Determinant is a number associated to a square matrix.

$\therefore$ (c) is the correct option.

(v) Here, $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$\therefore \quad A^2 = A \cdot A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0+1 & 0+0 \\ 0+0 & 1+0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore$ (b) is the correct option.

(vi) Here, $A = \begin{bmatrix} 1 & 2 \\ k & 6 \end{bmatrix}$

The inverse for the matrix does not exist if

$$|A| = 0$$

$$\Rightarrow \quad \begin{vmatrix} 1 & 2 \\ k & 6 \end{vmatrix} = 0 \quad \Rightarrow \quad 6 - 2k = 0$$

$$\Rightarrow \quad k = 3$$

$\therefore$ (b) is the correct option.

(vii) Here, the given curve is $x^2 + 3y + y^2 = 5$.

Differentiating with respect to x, we get

$$2x + 3\frac{dy}{dx} + 2y\cdot\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx}(3 + 2y) = -2x$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{2x}{(3+2y)}$$

Now, $$\left(\frac{dy}{dx}\right)_{(1,1)} = -\frac{2(1)}{3+2(1)} = -\frac{2}{5}$$

This is slope of tangent to the curve at (1, 1).

So, the slope of normal to the curve at (1, 1)

$$= -\frac{1}{\left(\frac{dy}{dx}\right)_{(1,\,1)}} = \frac{5}{2}.$$

$\therefore$ (b) is the correct option.

(viii) Let A be the area of circle of radius 'r'

$$\therefore \quad A = \pi r^2$$

The rate of change of area A with respect to its radius r

$$= \frac{dA}{dr}$$

$$= \frac{d}{dr}(\pi r^2) = 2\pi r$$

$$\therefore \quad \left(\frac{dA}{dr}\right)_{r=3} = 2\pi(3) = 6\pi$$

Hence, the area of circle is changing at the rate of 6π sq cm per cm.

$\therefore$ (a) is the correct option.

(ix) Here, $f(x) = x^2 + x + 1,\ x \in [0, 4]$

Since, Lagrange's mean value theorem holds for the function $f(x) = x^2 + x + 1,\ x \in [0, 4]$.

So, there exists a point $c \in (0, 4)$ such that

$$f'(c) = \frac{f(4) - f(0)}{4 - 0}$$

$$\Rightarrow \quad 2c + 1 = \frac{((4)^2 + 4 + 1) - (0 + 0 + 1)}{4 - 0}$$

$$\Rightarrow \quad 2c + 1 = 5 \quad \Rightarrow \quad 2c = 4$$

$$\Rightarrow \quad c = 2$$

$\therefore$ (b) is the correct option.

(x) Here, $P(A) = \frac{4}{5}$ and $P(A \cap B) = \frac{7}{10}$

$$\therefore \quad P(B|A) = \frac{P(A \cap B)}{P(A)}$$

$$= \frac{\left(\frac{7}{10}\right)}{\left(\frac{4}{5}\right)} = \frac{7}{8}$$

$\therefore$ (d) is the correct option.

(xi) Here, $a * b = \sqrt{a^2 + b^2}$

$$\therefore \quad 3 * 4 = \sqrt{(3)^2 + (4)^2}$$

$$= \sqrt{25} = 5$$

Hence, $3 * 4 = 5$.

(xii) Here, A is a square matrix of order 3, with $|A| = 4$

$$\therefore \quad |-2A| = (-2)^3|A| \qquad [\because |\lambda A| = \lambda^n \cdot |A|, \text{ where } n \text{ is order of matrix.}]$$

$$= -8 \cdot (4) = -32.$$

(xiii) Here, the differential equation is

$$\left(\frac{dy}{dx}\right)^5 + 3xy\left(\frac{d^3y}{dx^3}\right)^2 + y^2\left(\frac{d^2y}{dx^2}\right)^3 = 0$$

It involves the highest derivative of third order.

So it is of order 3.

It is a polynomial equation in $\frac{d^3y}{dx^3}, \frac{d^2y}{dx^2}$ and $\frac{dy}{dx}$.

So its degree can be defined.

The highest power of $\frac{d^3y}{dx^3}$ is 2. So it is of degree 2.

Hence, the sum of order and degree = 3 + 2 = 5.

(xiv) Let E and F be the events defined as follows:

E : A is selected.

F : B is selected.

$$\therefore \quad P(E) = \frac{4}{5}, \qquad P(F) = \frac{1}{3}$$

$$P(\overline{E}) = 1 - P(E), \qquad P(\overline{F}) = 1 - P(F)$$

$$= 1 - \frac{4}{5} = \frac{1}{5} \qquad = 1 - \frac{1}{3} = \frac{2}{3}$$

$\therefore$ P(none of them will be selected) = $P(\overline{E})\,P(\overline{F})$ [$\because$ Events E and F are independent $\Rightarrow \overline{E}$ and $\overline{F}$ are also in independent]

$$= \frac{1}{5} \times \frac{1}{3}$$

$$= \frac{2}{15}.$$

(xv) Here, mean = 5 $\Rightarrow$ $np = 5$ ...(1)

and variance = 4 $\Rightarrow$ $npq = 4$ ...(2)

Dividing (2) by (1), we get

$$\frac{npq}{np} = \frac{4}{5} \Rightarrow q = \frac{4}{5}$$

$$\Rightarrow 1 - p = \frac{4}{5} \quad [\because p + q = 1]$$

$$\Rightarrow p = 1 - \frac{4}{5}$$

$$\Rightarrow p = \frac{1}{5}$$

Putting $p = \frac{1}{5}$ in (1), we get $n = 25$

Hence, the number of events = 25.

Solution 2

(a) For $x \neq 3$, we have $f(x) = \frac{x^2 - 9}{x - 3} = \frac{(x+3)(x-3)}{x-3} = x + 3$

Now, $$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (x + 3)$$

$$= \lim_{h \to 0} [(3 - h) + 3] = 6$$

Also, $$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (x + 3)$$

$$= \lim_{h \to 0} [(3 + h) + 3] = 6$$

So, f is continuous at $x = 3$, if

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = f(3)$$

i.e., $6 = 6 = f(3)$

i.e., $f(3) = 6$.

OR

(b) Here, $x = at^2$

Differentiating both sides with respect to t, we get

$$\frac{dx}{dt} = \frac{d}{dt}(at^2) = a\frac{d}{dt}(t^2)$$

$$= 2at.$$

Also, $y = 2at$

Differentiating both sides with respect to t, we get

$$\frac{dy}{dt} = \frac{d}{dt}(2at) = 2a\frac{d}{dt}(t)$$

$$= 2a.$$

Hence, $$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{2a}{2at} = \frac{1}{t}.$$

Solution 3

Here, $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$ and $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = 2x^2 + 1$.

Then, $f \circ g : \mathbb{R} \to \mathbb{R}$ such that

$$(f \circ g)(x) = f(g(x)) = f(2x^2 + 1) = (2x^2 + 1)^3$$

Also, $g \circ f : \mathbb{R} \to \mathbb{R}$ such that

$$(g \circ f)(x) = g(f(x)) = g(x^3) = 2(x^3)^2 + 1 = 2x^6 + 1.$$

Solution 4

Here, $\lim_{x\to 0}\left[\frac{8^x - 4^x}{4x}\right]$ $\left[\frac{0}{0} \text{ form}\right]$

$= \lim_{x\to 0}\left[\frac{(8^x - 4^x)'}{(4x)'}\right]$ [By L' Hôpital's Rule]

$= \lim_{x\to 0}\left[\frac{8^x \log 8 - 4^x \log 4}{4}\right] = \frac{1}{4}(\log 8 - \log 4) = \frac{1}{4}\log 2.$

Solution 5

(a) Here, $$\int \frac{\sec^2 x}{\text{cosec}^2 x}dx = \int \tan^2 x\, dx$$

$$= \int (\sec^2 x - 1)dx$$

$$= \tan x - x + C.$$

OR

(b) Here, $$I = \int_0^{\pi/2} \frac{\sin^{3/2} x}{\sin^{3/2} x + \cos^{3/2} x}dx \quad ...(1)$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \frac{\sin^{3/2}\left(\frac{\pi}{2} - x\right)}{\sin^{3/2}\left(\frac{\pi}{2} - x\right) + \cos^{3/2}\left(\frac{\pi}{2} - x\right)}dx$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \frac{\cos^{3/2} x}{\cos^{3/2} x + \sin^{3/2} x}dx \quad ...(2)$$

Adding (1) and (2), we get

$$2I = \int_0^{\pi/2} \frac{\sin^{3/2} x}{\sin^{3/2} x + \cos^{3/2} x} dx + \int_0^{\pi/2} \frac{\cos^{3/2} x}{\cos^{3/2} x + \sin^{3/2} x} dx$$

$$= \int_0^{\pi/2} \frac{\sin^{3/2} x + \cos^{3/2} x}{\sin^{3/2} x + \cos^{3/2} x} dx$$

$$= \int_0^{\pi/2} 1\, dx$$

$$\Rightarrow \quad 2I = [x]_0^{\pi/2} = \frac{\pi}{2}$$

$$\therefore \quad I = \frac{\pi}{4}.$$

Solution 6

Here, $$\frac{dy}{dx} = 1 - xy + y - x$$

$$\Rightarrow \quad \frac{dy}{dx} = (1 - x) + y(1 - x)$$

$$\Rightarrow \quad \frac{dy}{dx} = (1 - x)(1 + y)$$

$$\Rightarrow \quad \frac{1}{1+y} dy = (1 - x)dx$$

Integrating both sides, we get

$$\int \frac{1}{1+y} dy = \int (1 - x) dx$$

$$\Rightarrow \quad \log|1 + y| = x - \frac{x^2}{2} + C$$

which is the required solution.

Solution 7

Here, $$\tan^{-1}\frac{1}{2} = \frac{\pi}{4} - \frac{1}{2}\cos^{-1}\frac{4}{5}$$

or $$\cos^{-1}\frac{4}{5} + 2\tan^{-1}\frac{1}{2} = \frac{\pi}{2}$$

Now, L.H.S. $= \cos^{-1}\frac{4}{5} + 2\tan^{-1}\frac{1}{2} = \tan^{-1}\left[\frac{\sqrt{1-\left(\frac{4}{5}\right)^2}}{\left(\frac{4}{5}\right)}\right] + \tan^{-1}\left[\frac{2\left(\frac{1}{2}\right)}{1-\left(\frac{1}{2}\right)^2}\right]$

$$\left[\because \cos^{-1}x = \tan^{-1}\frac{\sqrt{1-x^2}}{x} \text{ and } 2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}\right]$$

$$= \tan^{-1}\left(\frac{3}{5} \times \frac{5}{4}\right) + \tan^{-1}\frac{4}{3}$$

$$= \tan^{-1}\frac{3}{4} + \tan^{-1}\frac{4}{3}$$

$$= \tan^{-1}\frac{3}{4} + \cot^{-1}\frac{3}{4} = \frac{\pi}{2} = \text{R.H.S.}$$

Solution 8

Here, $$x = \tan\left(\frac{1}{a}\log y\right)$$

$$\Rightarrow \quad \tan^{-1}x = \frac{1}{a}\log y$$

$$\Rightarrow \quad a\tan^{-1}x = \log y$$

Differentiating both sides with respect to x, we get

$$\frac{a}{1+x^2} = \frac{1}{y}\frac{dy}{dx}$$

$$\Rightarrow \quad (1+x^2)\frac{dy}{dx} = ay$$

Differentiating both sides with respect to x, we get

$$(1+x^2)\frac{d^2y}{dx^2} + \frac{dy}{dx}(2x) = a\frac{dy}{dx}$$

$$\Rightarrow \quad (1+x^2)\frac{d^2y}{dx^2} + (2x-a)\frac{dy}{dx} = 0.$$

Solution 9

(a) Here, $$|x+3| = \begin{cases} x+3 & \text{if } x+3 \ge 0 \\ -(x+3) & \text{if } x+3 < 0 \end{cases} = \begin{cases} x+3 & \text{if } x \ge -3 \\ -(x+3) & \text{if } x < -3 \end{cases}$$

$\therefore$ The integral $\int_{-6}^{3}|x+3|\,dx$

$$= -\int_{-6}^{-3}(x+3)\,dx + \int_{-3}^{3}(x+3)\,dx$$

$$= -\left[\frac{x^2}{2}+3x\right]_{-6}^{-3} + \left[\frac{x^2}{2}+3x\right]_{-3}^{3}$$

$$= -\left[\left(\frac{9}{2}-9\right)-(18-18)\right] + \left[\left(\frac{9}{2}+9\right)-\left(\frac{9}{2}-9\right)\right]$$

$$= -\left[-\frac{9}{2}\right] + \left[\frac{27}{2}-\left(-\frac{9}{2}\right)\right]$$

$$= \frac{9}{2}+\frac{36}{2} = \frac{45}{2}.$$

OR

(b) Here, $$I = \int_0^{\pi}\frac{x\tan x}{\sec x+\tan x}dx = \int_0^{\pi}\frac{x\frac{\sin x}{\cos x}}{\frac{1}{\cos x}+\frac{\sin x}{\cos x}}dx$$

$$\Rightarrow \quad I = \int_0^{\pi}\frac{x\sin x}{1+\sin x}dx \qquad \text{...(1)}$$

$$\Rightarrow \qquad I = \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1 + \sin(\pi - x)}\, dx$$

$$\Rightarrow \qquad I = \int_0^{\pi} \frac{(\pi - x)\sin x}{1 + \sin x} \qquad \text{...(2)}$$

Adding (1) and (2), we get

$$2I = \int_0^{\pi} \frac{x \sin x}{1 + \sin x}\, dx + \int_0^{\pi} \frac{(\pi - x)\sin x}{1 + \sin x}\, dx$$

$$= \pi \int_0^{\pi} \frac{\sin x}{1 + \sin x}\, dx$$

$$= \pi \int_0^{\pi} \frac{\sin x}{1 + \sin x} \cdot \frac{1 - \sin x}{1 - \sin x}\, dx$$

$$= \pi \int_0^{\pi} \frac{\sin x - \sin^2 x}{1 - \sin^2 x}\, dx$$

$$= \pi \int_0^{\pi} \frac{\sin x - \sin^2 x}{\cos^2 x}\, dx$$

$$= \pi \int_0^{\pi} (\sec x \tan x - \tan^2 x)\, dx$$

$$= \pi \int_0^{\pi} (\sec x \tan x - \sec^2 x + 1)\, dx$$

$$= \pi \left[\sec x - \tan x + x\right]_0^{\pi}$$

$$= \pi[(-1 - 0 + \pi) - (1 - 0 + 0)]$$

$$\Rightarrow \qquad 2I = \pi(\pi - 2)$$

$$\therefore \qquad I = \frac{\pi}{2}(\pi - 2) = \frac{\pi^2}{2} - \pi.$$

Solution 10

(a) Here, Bag A contains 4 white and 3 black balls, and
Bag B contains 3 white and 5 black balls.

Let B_1, B_2 and B_3 be the events defined as follows:

B_1 : Two white balls are transferred from Bag A to Bag B.

B_2 : Two black balls are transferred from Bag A to Bag B.

B_3 : One white and one black ball are transferred from Bag A to Bag B.

E : A white ball is drawn from Bag B.

Then, $$P(B_1) = \frac{{}^4C_2}{{}^7C_2} = \frac{6}{21} = \frac{2}{7}, \qquad P(B_2) = \frac{{}^3C_2}{{}^7C_2} = \frac{3}{21} = \frac{1}{7},$$

$$P(B_3) = \frac{{}^4C_1 \times {}^3C_1}{{}^7C_2} = \frac{4 \times 3}{21} = \frac{4}{7}$$

After occurrence of event B_1, we have 5 white and 5 black balls in Bag B.

$\therefore \qquad P(E|B_1) = \dfrac{^5C_1}{^{10}C_1} = \dfrac{5}{10} = \dfrac{1}{2}$

After the occurrence of event B_2, we have 3 white and 7 black balls in Bag B.

$\therefore \qquad P(E|B_2) = \dfrac{^3C_1}{^{10}C_1} = \dfrac{3}{10}$

After the occurrence of event B_3, we have 4 white and 6 black balls in Bag B.

$\therefore \qquad P(E|B_3) = \dfrac{^4C_1}{^{10}C_1} = \dfrac{4}{10} = \dfrac{2}{5}$

So, by the law of total probability, we have

$$P(E) = P(B_1)\cdot P(E|B_1) + P(B_2)\cdot P(E|B_2) + P(B_3)\cdot P(E|B_3)$$

$$= \left(\frac{2}{7}\times\frac{1}{2}\right)+\left(\frac{1}{7}\times\frac{3}{10}\right)+\left(\frac{4}{7}\times\frac{2}{5}\right) = \frac{29}{70}$$

Hence, required probability $= \dfrac{29}{70}$.

OR

(b) Let $B(n, p)$ be binomial distribution with mean 9 and standard deviation $\dfrac{3}{2}$.

Then, $\qquad np = 9 \qquad$ ...(1)

and $\qquad \sqrt{npq} = \dfrac{3}{2} \quad$ or $\quad npq = \dfrac{9}{4} \qquad$ ...(2)

Dividing (2) by (1), we get $q = \dfrac{1}{4}$.

Also, $\qquad p = 1 - q = 1 - \dfrac{1}{4} = \dfrac{3}{4}$

Putting value of p in (1), we get $n = 12$

$\therefore$ The binomial distribution is $B\left(12, \dfrac{3}{4}\right)$.

Let X be a random variable defined as the number of successes

$\therefore$ Probability function, $P(X = x) = {^nC_x}\, p^x q^{n-x}$

$\therefore$ Required probability, $P(X \le 1) = P(X = 0) + P(X = 1)$

$$= {^{12}C_0}\left(\frac{3}{4}\right)^0\left(\frac{1}{4}\right)^{12} + {^{12}C_1}\left(\frac{3}{4}\right)^1\left(\frac{1}{4}\right)^{11}$$

$$= \left(\frac{1}{4}\right)^{12} + 36\left(\frac{1}{4}\right)^{12}$$

$$= 37\left(\frac{1}{4}\right)^{12}.$$

Solution 11

(a) Here, the given system of equations is

$$2x - 3y + 5z = 11$$
$$3x + 2y - 4z = -5$$
$$x + y - 2z = -3$$

This system of equations can be written as

$$AX = B$$

where $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$

Now, $|A| = \begin{vmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{vmatrix} = 2(-4 + 4) + 3(-6 + 4) + 5(3 - 2)$

$$= -1 \neq 0.$$

As $|A| \neq 0$, A^{-1} exists and the given system of equations has a unique solution.

Let A_{ij} denote cofactor of a_{ij} in $A = [a_{ij}]$

So, the cofactors of elements of A are

$A_{11} = (-1)^{1+1}\begin{vmatrix} 2 & -4 \\ 1 & -2 \end{vmatrix} = 0,\quad A_{21} = (-1)^{2+1}\begin{vmatrix} -3 & 5 \\ 1 & -2 \end{vmatrix} = -1,\quad A_{31} = (-1)^{3+1}\begin{vmatrix} -3 & 5 \\ 2 & -4 \end{vmatrix} = 2,$

$A_{12} = (-1)^{1+2}\begin{vmatrix} 3 & -4 \\ 1 & -2 \end{vmatrix} = 2,\quad A_{22} = (-1)^{2+2}\begin{vmatrix} 2 & 5 \\ 1 & -2 \end{vmatrix} = -9,\quad A_{32} = (-1)^{3+2}\begin{vmatrix} 2 & 5 \\ 3 & -4 \end{vmatrix} = 23$

$A_{13} = (-1)^{1+3}\begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} = 1,\quad A_{23} = (-1)^{2+3}\begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} = -5\quad A_{33} = (-1)^{3+3}\begin{vmatrix} 2 & -3 \\ 3 & 2 \end{vmatrix} = 13$

$\therefore\quad \text{adj } A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}$

Now, $A^{-1} = \dfrac{1}{|A|}(\text{adj } A) = -\begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}$

$\therefore\quad X = A^{-1}B \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}\begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$\therefore\quad x = 1,\quad y = 2,\quad z = 3$

Hence, the required solution is $x = 1$, $y = 2$, $z = 3$.

OR

(b) Here, L.H.S. $= \begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix}$

$= \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & px^3 \\ y & y^2 & py^3 \\ z & z^2 & pz^3 \end{vmatrix}$

$= \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + xyz\begin{vmatrix} 1 & x & px^2 \\ 1 & y & py^2 \\ 1 & z & pz^2 \end{vmatrix}$ [On taking x common from R_1, y common from R_2 and z common from R_3 of second determinant]

$= \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + pxyz\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$ [On taking p common from C_3 of second determinant]

$= \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} - pxyz\begin{vmatrix} x & 1 & x^2 \\ y & 1 & y^2 \\ z & 1 & z^2 \end{vmatrix}$ [Applying $C_1 \leftrightarrow C_2$ in second determinant]

$= \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + pxyz\begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix}$ [Applying $C_2 \leftrightarrow C_3$ in second determinant]

$= (1 + pxyz)\begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix}$

$= (1 + pxyz)\begin{vmatrix} x-y & x^2-y^2 & 0 \\ y-z & y^2-z^2 & 0 \\ z & z^2 & 1 \end{vmatrix}$ [Applying $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$]

$= (1 + pxyz)(x - y)(y - z)\begin{vmatrix} 1 & x+y & 0 \\ 1 & y+z & 0 \\ z & z^2 & 1 \end{vmatrix}$ [On taking $(x - y)$ common from R_1 and $(y - z)$ on expanding along C_3 from R_2]

$= (1 + pxyz)(x - y)(y - z)[0 - 0 + 1\{(y + z) - (x + y)\}]$ [Expanding along C_3]

$= (1 + pxyz)(x - y)(y - z)(z - x) =$ R.H.S.

Solution 12

(a) Let $I = \int \tan^{-1}\sqrt{x}\, dx$.

Substituting $\tan^{-1}\sqrt{x} = y \Rightarrow \sqrt{x} = \tan y$

$\Rightarrow x = \tan^2 y$

$\Rightarrow dx = 2\tan y \sec^2 y\, dy$

$\therefore$

$$I = 2\int y \cdot \tan y \sec^2 y\, dy$$

$$= 2\left[y \cdot \left(\frac{\tan^2 y}{2}\right) - \int 1 \cdot \left(\frac{\tan^2 y}{2}\right) dy\right] \quad \text{[Integrating by Parts]}$$

$$= y \cdot \tan^2 y - \int \tan^2 y\, dy$$

$$= y \cdot \tan^2 y - \int (\sec^2 y - 1)\, dy$$

$$= y \tan^2 y - \tan y + y + C$$

$$= \left(\tan^{-1}\sqrt{x}\right)x - \sqrt{x} + \tan^{-1}\sqrt{x} + C$$

$$= (x+1)\tan^{-1}\sqrt{x} - \sqrt{x} + C.$$

OR

(b) Here,
$$\int \frac{3x+5}{x^3 - x^2 - x + 1}\, dx = \int \frac{3x+5}{(x-1)^2(x+1)}\, dx \quad \text{...(1)}$$

Let
$$\frac{3x+5}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{D}{x+1} \quad \text{...(2)}$$

$\Rightarrow \quad 3x + 5 = A(x-1)(x+1) + B(x+1) + D(x-1)^2$

$\Rightarrow \quad 3x + 5 = A(x^2 - 1) + B(x+1) + D(x^2 + 1 - 2x)$

$\Rightarrow \quad 3x + 5 = (A + D)x^2 + (B - 2D)x + (-A + B + D)$

Comparing coefficients of x^2, x and constant terms both sides, we get

$$A + D = 0, \quad B - 2D = 3, \quad -A + B + D = 5$$

Solving these equations, we get

$$A = -\frac{1}{2}, \qquad B = 4, \qquad D = \frac{1}{2}$$

Putting values of A, B and D in (2), we get

$$\frac{3x+5}{(x-1)^2(x+1)} = -\frac{1}{2(x-1)} + \frac{4}{(x-1)^2} + \frac{1}{2(x+1)} \quad \text{...(3)}$$

From (1) and (3), we get

$$\int \frac{3x+5}{x^3 - x^2 - x + 1}\, dx = -\frac{1}{2}\int \frac{1}{x-1}\, dx + 4\int \frac{1}{(x-1)^2}\, dx + \frac{1}{2}\int \frac{1}{x+1}\, dx$$

$$= -\frac{1}{2}\log|x-1| - \frac{4}{(x-1)} + \frac{1}{2}\log|x+1| + C$$

$$= \frac{1}{2}\log\left|\frac{x+1}{x-1}\right| - \frac{4}{x-1} + C.$$

Solution 13

Let r be the radius of base and h the height of cone.

Let x be the distance between the centre of sphere and the centre of the base of cone.

Then, $x = h - 12 \Rightarrow x^2 = (h-12)^2$

$\Rightarrow x^2 = h^2 + 144 - 24h$ ...(1)

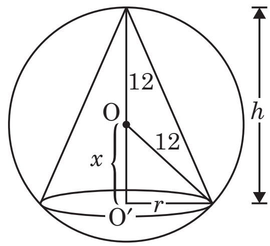
and $r^2 = (12)^2 - x^2 \Rightarrow r^2 = 144 - (h^2 + 144 - 24h)$ [Using (1)]

$\Rightarrow r^2 = 24h - h^2$ ...(2)

$\therefore$ Volume of cone, $V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi(24h - h^2)h$ [Using (2)]

$= \frac{1}{3}\pi(24h^2 - h^3)$



$\therefore \frac{dV}{dh} = \frac{1}{3}\pi(48h - 3h^2)$

Now, $\frac{dV}{dh} = 0 \Rightarrow \frac{1}{3}\pi(48h - 3h^2) = 0$

$\Rightarrow \frac{1}{3}\pi(48 - 3h) = 0$

$\Rightarrow 48 - 3h = 0$ [$\because h \neq 0$]

$\Rightarrow h = 16$

Also, $\frac{d^2V}{dh^2} = \frac{1}{3}\pi(48 - 6h) \Rightarrow \left(\frac{d^2V}{dh^2}\right)_{h=16} = \frac{1}{3}\pi(48 - 96) = -16\pi < 0$

So, by second derivative test volume is maximum when $h = 16$.

Hence, the required height is 16 cm.

Solution 14

Here, Box A contains 2 gold and 1 silver coin

Box B contains 1 gold and 2 silver coins

Box C contains 3 silver coins

Let B_1, B_2, B_3 and E be the events defined as follows:

B_1 : Box A is selected.

B_2 : Box B is selected.

B_3 : Box C is selected.

E : Silver coin is drawn.

Then, $P(B_1) = P(B_2) = P(B_3) = \frac{1}{3}$

and $P(E|B_1) = \frac{1}{3}$, $P(E|B_2) = \frac{2}{3}$, $P(E|B_3) = \frac{3}{3}$

One coin is of silver and other two coins are also of silver (i.e., all the three coins are of silver) means Box C.

$\therefore$ By Bayes' Theorem, we have

$$P(B_3|E) = \frac{P(B_3)P(E\,|\,B_3)}{P(B_1)P(E\,|\,B_1)+P(B_2)P(E\,|\,B_2)+P(B_3)P(E\,|\,B_3)}$$

$$= \frac{\left(\frac{1}{3}\right)\left(\frac{3}{3}\right)}{\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)+\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)+\left(\frac{1}{3}\right)\left(\frac{3}{3}\right)} = \frac{1}{2}$$

Hence, required probability = $\frac{1}{2}$.

SECTION-B

Solution 15

(i) Let $\overrightarrow{a_1} = a\hat{i}+3\hat{j}-2\hat{k}$ and $\overrightarrow{a_2} = 3\hat{i}-4\hat{j}+b\hat{k}$

Vectors $\overrightarrow{a_1}$ and $\overrightarrow{a_2}$ are collinear if

$\overrightarrow{a_1} = \lambda\overrightarrow{a_2}$, where λ is a scalar

$\therefore$ $a\hat{i}+3\hat{j}-2\hat{k} = \lambda(3\hat{i}-4\hat{j}+b\hat{k})$

Comparing coefficients of $\hat{i}$, $\hat{j}$ and $\hat{k}$, we get

$a = 3\lambda$, ...(1), $3 = -4\lambda$, ...(2), $-2 = b\lambda$, ...(3)

From (2), we get $\lambda = -\frac{3}{4}$

Putting this value of λ in (1) and (3), we get

$$a = -\frac{9}{4} \text{ and } b = \frac{8}{3}$$

So, $(a, b) = \left(-\frac{9}{4}, \frac{8}{3}\right)$

$\therefore$ (b) is the correct option.

(ii) Here, $3x - 2y + 4z = 12$

$\Rightarrow$ $\frac{3x}{12}-\frac{2y}{12}+\frac{4z}{12} = 1$

$\Rightarrow$ $\frac{x}{4}+\frac{y}{(-6)}+\frac{z}{3} = 1$

So, the intercepts made by the plane on coordinate axes are 4, –6, 3.

$\therefore$ (b) is the correct option.

(iii) Here, $\vec{a} = 6\hat{i}+2\hat{j}+3\hat{k}$ and $\vec{b} = 2\hat{i}-9\hat{j}+6\hat{k}$

$\therefore$ $\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$, where θ is the angle between $\vec{a}$ and $\vec{b}$.

$$= \frac{(6\hat{i}+2\hat{j}+3\hat{k})\cdot(2\hat{i}-9\hat{j}+6\hat{k})}{\sqrt{(6)^2+(2)^2+(3)^2}\cdot\sqrt{(2)^2+(-9)^2+(6)^2}}$$

$$= \frac{6(2) + 2(-9) + 3(6)}{\sqrt{49} \cdot \sqrt{121}}$$

$$= \frac{12}{(7)(11)} = \frac{12}{77}$$

$$\Rightarrow \qquad \theta = \cos^{-1}\left(\frac{12}{77}\right).$$

(iv) Let $\vec{a} = 2\hat{i} + \hat{j} - \hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{c} = -3\hat{i} - \hat{j} + \hat{k}$.

Volume of the parallelepiped whose co-terminus edges are $\vec{a}$, $\vec{b}$ and $\vec{c}$

$$= \vec{a} \cdot (\vec{b} \times \vec{c})$$

$$= \begin{vmatrix} 2 & 1 & -1 \\ 1 & 2 & 3 \\ -3 & -1 & 1 \end{vmatrix}$$

$$= 2(2 + 3) - 1(1 + 9) - 1(-1 + 6) = -5$$

So, the volume of parallelepiped = |–5| = 5 cubic units.

(v) Equation of plane, perpendicular to line whose direction ratios are 3, 1, 5 is

$$3x + y + 5z = \lambda \qquad ...(1)$$

Since, the plane passes through (–2, 1, 3)

$$\therefore \qquad 3(-2) + 1 + 5(3) = \lambda$$

$$\Rightarrow \qquad \lambda = 10$$

Putting $\lambda = 10$ in (1), we get required equation of plane as $3x + y + 5z = 10$.

Solution 16

(a) Here, $\vec{a}$ and $\vec{b}$ are perpendicular vectors

$$\therefore \qquad \vec{a} \cdot \vec{b} = 0$$

Also, $\qquad |\vec{a} + \vec{b}| = 13$ and $|\vec{a}| = 5$

Now, $\qquad |\vec{a} + \vec{b}|^2 = (\vec{a} + \vec{b})^2 = (\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b})$

$$= \vec{a} \cdot \vec{a} + \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{a} + \vec{b} \cdot \vec{b}$$

$$= |\vec{a}|^2 + 2\vec{a} \cdot \vec{b} + |\vec{b}|^2 \qquad [\because \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}]$$

$$\Rightarrow \qquad |\vec{a} + \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 \qquad [\because \vec{a} \cdot \vec{b} = 0]$$

$$\Rightarrow \qquad (13)^2 = (5)^2 + |\vec{b}|^2$$

$$\Rightarrow \qquad 169 = 25 + |\vec{b}|^2$$

$$\Rightarrow \qquad |\vec{b}|^2 = 144 \quad \Rightarrow \quad |\vec{b}| = 12.$$

OR

(b) Here, the projection of $\vec{a} = \lambda\hat{i} + \hat{j} + 4\hat{k}$ on $\vec{b} = 2\hat{i} + 6\hat{j} + 3\hat{k}$ is 4 units.

i.e., $$\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|} = 4$$

$$\Rightarrow \quad \frac{(\lambda\hat{i}+\hat{j}+4\hat{k})\cdot(2\hat{i}+6\hat{j}+3\hat{k})}{|2\hat{i}+6\hat{j}+3\hat{k}|} = 4$$

$$\Rightarrow \quad \frac{2\lambda+6+12}{\sqrt{(2)^2+(6)^2+(3)^2}} = 4 \quad \Rightarrow \quad \frac{2\lambda+18}{7} = 4$$

$$\Rightarrow \quad \lambda = 5.$$

Solution 17

(a) Here, the given planes are

$$2x + 2y - 3z - 7 = 0$$

$$2x + 5y + 3z - 9 = 0$$

Any plane passing through the intersection of the given planes is

$$(2x + 2y - 3z - 7) + \lambda(2x + 5y + 3z - 9) = 0 \qquad \text{...(1)}$$

$$\Rightarrow \quad (2 + 2\lambda)x + (2 + 5\lambda)y + (-3 + 3\lambda)z = 7 + 9\lambda$$

$$\Rightarrow \quad \frac{x}{\left(\frac{7+9\lambda}{2+2\lambda}\right)} + \frac{y}{\left(\frac{7+9\lambda}{2+5\lambda}\right)} + \frac{z}{\left(\frac{7+9\lambda}{3\lambda-3}\right)} = 1$$

∴ Intercepts on x-axis and z-axis are $\left(\frac{7+9\lambda}{2+2\lambda}\right)$ and $\left(\frac{7+9\lambda}{3\lambda-3}\right)$ respectively.

Since, the intercepts made by the required plane on the x-axis and z-axis are equal.

$$\therefore \quad \frac{7+9\lambda}{2+2\lambda} = \frac{7+9\lambda}{3\lambda-3}$$

$$\Rightarrow \quad (7 + 9\lambda)(3\lambda - 3) = (7 + 9\lambda)(2 + 2\lambda)$$

$$\Rightarrow \quad (7 + 9\lambda)\{(3\lambda - 3) - (2 + 2\lambda)\} = 0$$

$$\Rightarrow \quad (7 + 9\lambda)(\lambda - 5) = 0$$

$$\Rightarrow \quad \lambda = -\frac{7}{9}, 5$$

Putting $\lambda = -\frac{7}{9}$ in (1), we get

$$(2x + 2y - 3z - 7) - \frac{7}{9}(2x + 5y + 3z - 9) = 0$$

$$\Rightarrow \quad 4x - 17y - 48z = 0$$

Putting $\lambda = 5$ in (1), we get

$$(2x + 2y - 3z - 7) + 5(2x + 5y + 3z - 9) = 0$$

$$\Rightarrow \quad 12x + 27y + 12z - 52 = 0$$

Hence, the equations of the required planes are

$$4x - 17y - 48z = 0 \quad \text{and} \quad 12x + 27y + 12z - 52 = 0.$$

OR

(b) Let the equation of the required line through (2, 1, 3) be

$$\frac{x-2}{a} = \frac{y-1}{b} = \frac{z-3}{c} \qquad \text{...(1)}$$

where, a, b and c are direction ratios of line (1).

Since, line (1) is perpendicular to the given lines

$$\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{3} \text{ and } \frac{x}{-3} = \frac{y}{2} = \frac{z}{5}$$

$\therefore \quad 1 \cdot a + 2 \cdot b + 3 \cdot c = 0$ ...(2)

$-3 \cdot a + 2 \cdot b + 5 \cdot c = 0$ ...(3)

Solving (2) and (3) by cross multiplication, we get

$$\frac{a}{10-6} = \frac{b}{-9-5} = \frac{c}{2+6}$$

i.e., $\frac{a}{4} = \frac{b}{-14} = \frac{c}{8}$

i.e., $\frac{a}{2} = \frac{b}{-7} = \frac{c}{4} = \lambda$ (say)

$\therefore \quad a = 2\lambda, b = -7\lambda$ and $c = 4\lambda$

Substituting the values of a, b and c in (1), we get

$$\frac{x-2}{2\lambda} = \frac{y-1}{-7\lambda} = \frac{z-3}{4\lambda}$$

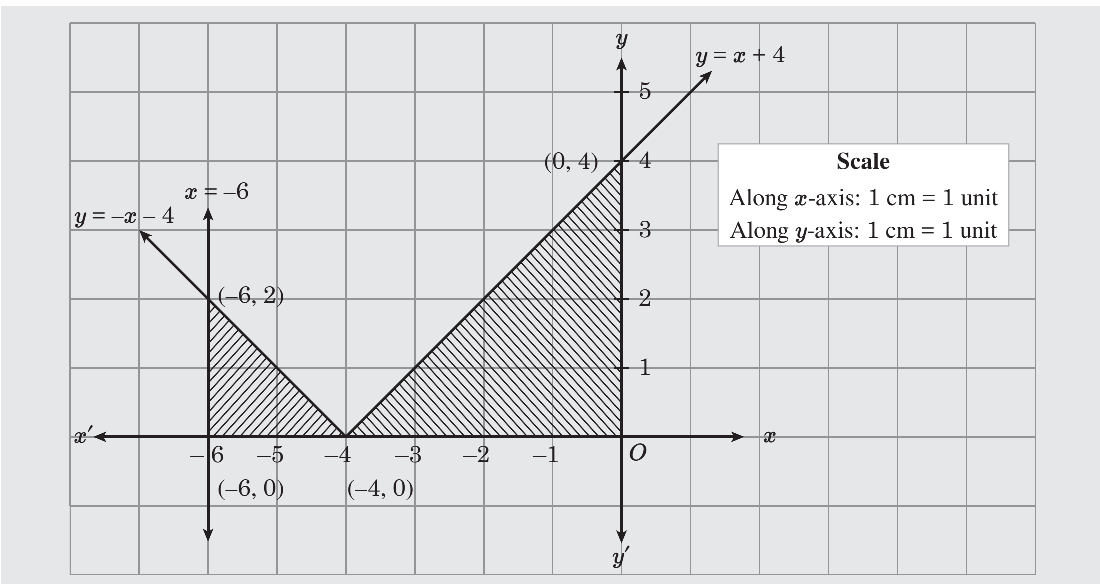
$\Rightarrow \quad \frac{x-2}{2} = \frac{y-1}{-7} = \frac{z-3}{4}$

which is the required equation of line.

Solution 18

Here, the graph of $y = |x + 4|$ can be obtained from graph of $y = |x|$ by shifting the graph horizontally 4 units to the left for the graph $y = |x|$.

$$y = |x+4|, \text{ i.e., } y = \begin{cases} x+4, & \text{if } x \geq -4 \\ -x-4, & \text{if } x \leq -4 \end{cases}$$



Also, the curve $x = -6$ is a straight line passing through (–6, 0) and parallel to y-axis and the curve $x = 0$ is y-axis.

$\therefore$ Required area $= \int_{-6}^{0} |x+4|\,dx$

$$= \int_{-6}^{-4} [(-x-4)-0]\,dx + \int_{-4}^{0} [(x+4)-0]\,dx$$

$$= \left[-\frac{x^2}{2} - 4x\right]_{-6}^{-4} + \left[\frac{x^2}{2} + 4x\right]_{-4}^{0}$$

$$= [(-8 + 16) - (-18 + 24)] + [(0 + 0) - (8 - 16)] = 10$$

Hence, required area is 10 sq units.

SECTION-C

Solution 19

(i) Here, $x = 100 - 4p \Rightarrow p = \dfrac{100-x}{4}$

$\therefore$ Revenue function, $R = px$

$$= \left(\frac{100-x}{4}\right)x = 25x - \frac{x^2}{4}$$

So, $\text{MR} = \dfrac{dR}{dx}$

$$= \frac{d}{dx}\left(25x - \frac{x^2}{4}\right) = 25 - \frac{x}{2}$$

Given, $\text{MR} = 0 \Rightarrow 25 - \dfrac{x}{2} = 0$

$\Rightarrow \dfrac{x}{2} = 25 \Rightarrow x = 50$

$\therefore$ (c) is the correct option.

(ii) Here, the lines of regression are parallel to coordinate axes, i.e., the lines intersect at right angles. So, coefficient of correlation $(r) = 0$.

$\therefore$ (b) is the correct option.

(iii) Here, $C(x) = \dfrac{x^3}{3} + 5x^2 - 16x + 2$

$\therefore$ Marginal cost function, $\text{MC} = \dfrac{dC}{dx}$

$$= \frac{d}{dx}\left(\frac{x^3}{3} + 5x^2 - 16x + 2\right)$$

$$= x^2 + 10x - 16.$$

(iv) Here, $\text{MR} = 11 - 3x + 4x^2$

Now, $\text{MR} = \dfrac{dR}{dx}$

$\Rightarrow dR = (\text{MR})dx$

$dR = (11 - 3x + 4x^2)dx$

Integrating, we get

$$\int dR = \int (11 - 3x + 4x^2)\, dx$$

$$\Rightarrow \quad R = 11x - \frac{3x^2}{2} + 4\frac{x^3}{3} + \lambda \qquad ...(1)$$

Now, when $x = 0$, $R = 0$

$$\therefore \quad 0 = 0 - 0 + 0 + \lambda \quad \Rightarrow \quad \lambda = 0$$

Putting $\lambda = 0$ in (1), we get revenue function as

$$R = 11x - \frac{3x^2}{2} + \frac{4}{3}x^3.$$

(v) Here, $b_{yx} = 1$ and $b_{xy} = \frac{1}{2}$

$$\therefore \quad \tan\theta = \left|\frac{b_{yx} \cdot b_{xy} - 1}{b_{yx} + b_{xy}}\right|$$, where θ is the angle between regression lines.

$$= \left|\frac{1\left(\frac{1}{2}\right) - 1}{1 + \frac{1}{2}}\right| = \left|\frac{-\frac{1}{2}}{\frac{3}{2}}\right| = \left|-\frac{1}{3}\right| = \frac{1}{3}$$

$$\Rightarrow \quad \theta = \tan^{-1}\left(\frac{1}{3}\right).$$

Solution 20

(a) Here, fixed cost $F =$ ₹ 24000, variable cost is 25% of the total revenue and sale price per unit is $p =$ ₹ 8.

$\therefore$ Revenue function, $R(x) = 8x$

Variable cost, $V(x) =$ 25% of $R(x)$

$=$ 25% of $(8x) = 2x$

$\therefore$ Cost function, $C(x) = F + V(x)$

$= 24000 + 2x$

Profit function, $P(x) = R(x) - C(x)$

$= 8x - (24000 + 2x)$

$= 6x - 24000$

At break even point, $P(x) = 0$

$\Rightarrow \quad 6x - 24000 = 0$

$\Rightarrow \quad x = 4000$

Hence, break even point is 4000.

OR

(b) Here, $C(x) = \frac{3}{4}x^2 - 7x + 27$

$$\therefore \quad \text{AC} = \frac{C}{x} = \frac{3}{4}x - 7 + \frac{27}{x} \text{ and MC} = \frac{dC}{dx} = \frac{3}{2}x - 7$$

Now, $$\text{AC} = \text{MC}$$

$$\Rightarrow \quad \frac{3}{4}x - 7 + \frac{27}{x} = \frac{3}{2}x - 7$$

$$\Rightarrow \quad \frac{27}{x} = \frac{3}{4}x$$

$$\Rightarrow \quad x^2 = 36 \quad \Rightarrow \quad x = 6 \qquad [\because x > 0]$$

Hence, the required level of output is 6 units.

Solution 21

We have the following table:

x_i	y_i	$u_i = x_i - 5$	$v_i = y_i - 10$	u_iv_i	u_i^2	v_i^2
1	4	–4	–6	24	16	36
2	8	–3	–2	6	9	4
3	2	–2	–8	16	4	64
4	12	–1	2	–2	1	4
5	10	0	0	0	0	0
6	14	1	4	4	1	16
7	16	2	6	12	4	36
8	6	3	–4	–12	9	16
9	18	4	8	32	16	64
$\Sigma x_i = 45$	$\Sigma y_i = 90$	$\Sigma u_i = 0$	$\Sigma v_i = 0$	$\Sigma u_iv_i = 80$	$\Sigma u_i^2 = 60$	$\Sigma v_i^2 = 240$

Here, $n = 9$.

The regression coefficient of y on x is given by

$$b_{yx} = b_{vu} = \frac{n\Sigma u_iv_i - (\Sigma u_i)(\Sigma v_i)}{n\Sigma u_i^2 - (\Sigma u_i)^2} = \frac{(9)(80) - (0)(0)}{(9)(60) - (0)^2} = \frac{4}{3}$$

We have, $\bar{x} = \frac{\Sigma x_i}{n} = \frac{45}{9} = 5$ and $\bar{y} = \frac{\Sigma y_i}{n} = \frac{90}{9} = 10$

Now, the regression equation of y on x is

$$y - 10 = \frac{4}{3}(x - 5)$$

$$\Rightarrow \quad y = \frac{4}{3}x + \frac{10}{3} \qquad \text{...(1)}$$

On putting $x = 14$ in (1), we get $y = 22$

Hence, the required value of y is 22.

Solution 22

(a) Let x be the number of units of product A and y be the number of units of product B.

Let gross income = ₹ Z.

We can represent the given L.P.P. in the following tabular form:

	Product *A*	**Product *B***	**Requirement**
Income (in ₹)	$48x$	$40y$	Maximise
Teakwood (running feet)	$2x$	y	At most 90
Plywood (running feet)	x	$2y$	At most 80
Rosewood (running feet)	x	y	At most 50

Hence, given L.P.P. is, Maximise $Z = 48x + 40y$

subject to the constraints:

$2x + y \le 90$, $x + 2y \le 80$, $x + y \le 50$, $x \ge 0$, $y \ge 0$

We consider the following equations:

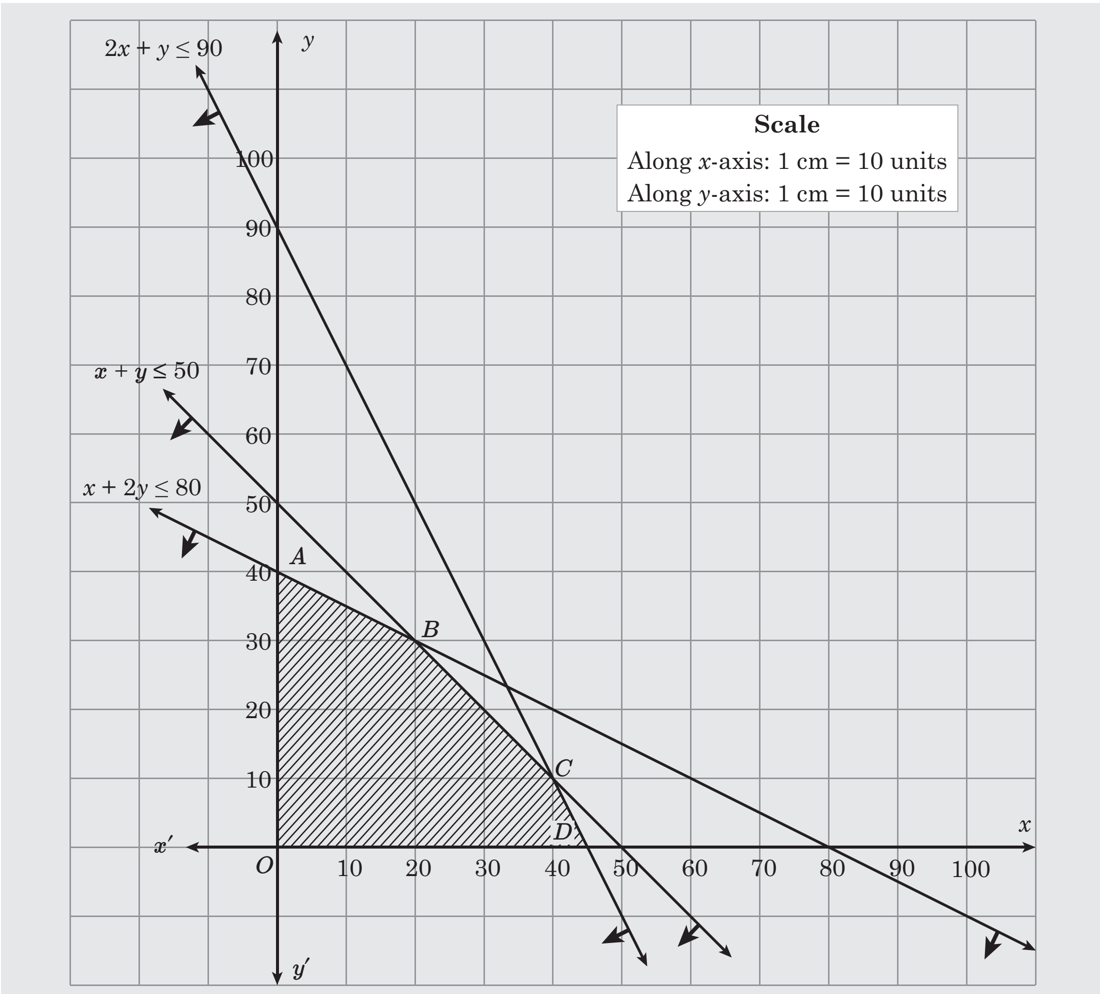
$2x + y = 90$, $x + 2y = 80$, $x + y = 50$, $x = 0$, $y = 0$

x	0	45
y	90	0

x	0	80
y	40	0

x	0	50
y	50	0

The feasible region of L.P.P. is bounded, as shown shaded in the graph.



Corner Points	Value of Z ($Z = 48x + 40y$)
$A(0, 40)$	$Z = 48(0) + 40(40) = 1600$
$B(20, 30)$	$Z = 48(20) + 40(30) = 2160$
$C(40, 10)$	$Z = 48(40) + 40(10) = 2320$
$D(45, 0)$	$Z = 48(45) + 40(0) = 2160$
$O(0, 0)$	$Z = 48(0) + 40(0) = 0.$

Since, the feasible region is bounded and 2320 is the maximum value of Z at corner points.

$\therefore$ 2320 is the maximum value of Z in the feasible region at $x = 40$, $y = 10$.

Hence, number of units of product $A = 40$, number of units of product $B = 10$ and maximum gross income = ₹ 2320.

OR

(b) Let x kg of product A and y kg of product B be produced.

Let total cost = ₹ Z.

We can represent the given L.P.P. in the following tabular form:

	Product A	Product B	Requirement
Cost (in ₹)	$20x$	$40y$	Minimise
Nutrient *P*	$36x$	$6y$	At least 108
Nutrient *Q*	$3x$	$12y$	At least 36
Nutrient *R*	$20x$	$10y$	At least 100

Hence, given L.P.P. is, Minimise $Z = 20x + 40y$

subject to the constraints:

$$36x + 6y \geq 108, \quad 3x + 12y \geq 36, \quad 20x + 10y \geq 100, \quad x \geq 0, \quad y \geq 0$$

We consider the following equations:

$36x + 6y = 108$, i.e., $6x + y = 18$

x	0	3
y	18	0

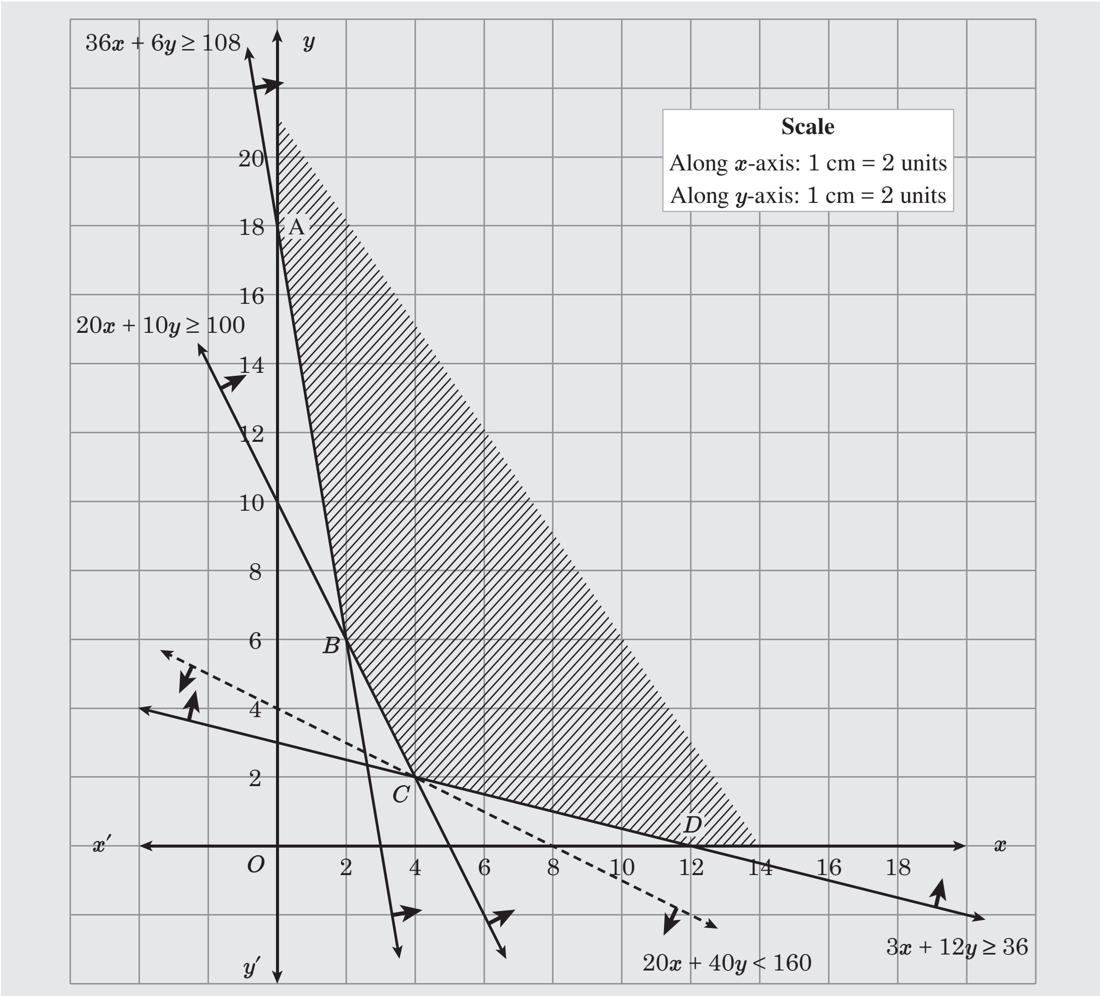
$3x + 12y = 36$, i.e., $x + 4y = 12$

x	0	12
y	3	0

$20x + 10y = 100$, i.e., $2x + y = 10$

x	0	5
y	10	0

$x = 0$, $y = 0$



Corner Points	Value of Z ($Z = 20x + 40y$)
$A(0, 18)$	$Z = 20(0) + 40(18) = 720$
$B(2, 6)$	$Z = 20(2) + 40(6) = 280$
$C(4, 2)$	$Z = 20(4) + 40(2) = 160$
$(12, 0)$	$Z = 20(12) + 4(0) = 240.$

Since, the feasible region is unbounded and 160 is the minimum value of Z at corner points.

So, we consider the open half plane $20x + 40y < 160$ which has no point in common with the feasible region.

$\therefore$ 160 is the minimum value of Z in the feasible region at $x = 4$, $y = 2$.

Hence, the number of kilograms of product $A = 4$, number of kilogram of product $B = 2$ and minimum cost = ₹ 160.