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Latest CBSE Sample Paper 2016-17

Time: 3 Hours **Max. Marks: 100**

General Instructions

(i) **All** questions are compulsory.

(ii) Please check that this question paper consists of **29** questions.

(iii) Questions **1** to **4** in Section A are Very Short Answer Type Questions carrying **1** mark each.

(iv) Questions **5** to **12** in Section B are Short Answer I Type Questions carrying **2** marks each.

(v) Questions **13** to **23** in Section C are Long Answer I Type Questions carrying **4** marks each.

(vi) Questions **24** to **29** in Section D are Long Answer II Type Questions carrying **6** marks each.

(vii) Please write down the serial number of the question before attempting it.

Section A

Question numbers **1** *to* **4** *carry* **1** *mark each.*

1. State the reason why the relation $R = \{(a, b) : a \leq b^2\}$ on the set $\mathbb{R}$ of real numbers is not reflexive.

2. If A is a square matrix of order 3 and $|2A| = k|A|$, then find the value of k.

3. If $\vec{a}$ and $\vec{b}$ are two non-zero vectors such that $|\vec{a} \times \vec{b}| = \vec{a} \cdot \vec{b}$, then find the angle between $\vec{a}$ and $\vec{b}$.

4. If $*$ is a binary operation on the set $\mathbb{R}$ of real numbers defined by $a * b = a + b - 2$, then find the identity element for the binary operation $*$.

Section B

Question numbers **5** *to* **12** *carry* **2** *marks each.*

5. Simplify: $\cot^{-1}\left(\dfrac{1}{\sqrt{x^2-1}}\right)$, for $x < -1$.

6. Prove that the diagonal elements of a skew symmetric matrix are all zeros.

7. If $y = \tan^{-1}\left(\dfrac{5x}{1-6x^2}\right)$, $-\dfrac{1}{\sqrt{6}} < x < \dfrac{1}{\sqrt{6}}$, then prove that

$$\frac{dy}{dx} = \frac{2}{1+4x^2} + \frac{3}{1+9x^2}.$$

8. If x changes from 4 to 4.01, then find the approximate change in $\log_e x$.

9. Find $\displaystyle\int \left(\frac{1-x}{1+x^2}\right)^2 e^x \, dx$.

10. Obtain the differential equation of the family of circles passing through the points $(a, 0)$ and $(-a, 0)$.

11. If $|\vec{a} + \vec{b}| = 60$, $|\vec{a} - \vec{b}| = 40$ and $|\vec{a}| = 22$, then find $|\vec{b}|$.

12. If $P(A) = \dfrac{2}{5}$, $P(B) = \dfrac{1}{3}$, $P(A \cap B) = \dfrac{1}{5}$, then find $P(\overline{A}|\overline{B})$.

Section C

*Question numbers **13** to **23** carry **4** marks each.*

13. If $A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$, then using A^{-1}, solve the following system of equations:

$$x - 2y = -1, \;\; 2x + y = 2.$$

14. Discuss the differentiability of the function $f(x) = \begin{cases} 2x - 1 & \text{if } x < \frac{1}{2} \\ 3 - 6x & \text{if } x \geq \frac{1}{2} \end{cases}$ at $x = \frac{1}{2}$.

OR

For what value of k is the following function continuous at $x = -\frac{\pi}{6}$?

$$f(x) = \begin{cases} \dfrac{\sqrt{3}\ \sin x + \cos x}{x + \frac{\pi}{6}} & \text{if } x \neq -\frac{\pi}{6} \\ k & \text{if } x = -\frac{\pi}{6} \end{cases}.$$

15. If $x = a \sin pt$, $y = b \cos pt$, then show that $(a^2 - b^2)y \dfrac{d^2y}{dx^2} + b^2 = 0$.

16. Find the equation of the normal to the curve $2y = x^2$,which passes through the point $(2, 1)$.

OR

Separate the interval $\left[0, \frac{\pi}{2}\right]$ into subintervals in which the function $f(x) = \sin^4 x + \cos^4 x$ is strictly increasing or strictly decreasing.

17. A magazine seller has 500 subscribers and collects annual subscription charges of ₹ 300 per subscriber. She proposes to increase the annual subscription charges and it is believed that for every increase of ₹ 1, one subscriber will discontinue. What increase will bring maximum income to her? Make appropriate assumptions in order to apply derivatives to reach the solution. **Write one important role of magazines in our lives.**

18. Find $\displaystyle\int \frac{\sin x}{(\cos^2 x + 1)(\cos^2 x + 4)}\, dx$.

19. Find the general solution of the differential equation $(1 + \tan y)(dx - dy) + 2x\, dy = 0$.

OR

Solve the following differential equation:

$$\left(1 + e^{x/y}\right) dx + e^{x/y} \left(1 - \frac{x}{y}\right) dy = 0.$$

20. Prove that: $\vec{a} \cdot \{(\vec{b} + \vec{c}) \times (\vec{a} + 2\vec{b} + 3\vec{c})\} = [\, \vec{a}\ \vec{b}\ \vec{c}\,]$.

21. Find the values of a so that the following lines are skew:

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-a}{4}, \;\; \frac{x-4}{5} = \frac{y-1}{2} = z.$$

22. A bag contains 4 green and 6 white balls. Two balls are drawn one by one without replacement. If the second ball drawn is white, what is the probability that the first ball drawn is also white?

23. Two cards are drawn successively with replacement from a well shuffled pack of 52 cards. Find the probability distribution of the number of diamond cards drawn. Also, find the mean and the variance of the distribution.

Section D

Question numbers **24** *to* **29** *carry* **6** *marks each.*

24. Let $f : [0, \infty) \to \mathbb{R}$ be a function defined by $f(x) = 9x^2 + 6x - 5$. Prove that f is not invertible. Modify, only the codomain of f to make f invertible and then find its inverse.

OR

Let $*$ be a binary operation defined on $\mathbb{Q} \times \mathbb{Q}$ by $(a, b) * (c, d) = (ac, b + ad)$, where $\mathbb{Q}$ is the set of rational numbers. Determine, whether $*$ is commutative and associative. Find the identity element for $*$ and the invertible elements of $\mathbb{Q} \times \mathbb{Q}$.

25. Using properties of determinants, prove that $\begin{vmatrix} \dfrac{(a+b)^2}{c} & c & c \\ a & \dfrac{(b+c)^2}{a} & a \\ b & b & \dfrac{(c+a)^2}{b} \end{vmatrix} = 2(a + b + c)^3.$

OR

If $p \neq 0$, $q \neq 0$ and $\begin{vmatrix} p & q & p\alpha + q \\ q & r & q\alpha + r \\ p\alpha + q & q\alpha + r & 0 \end{vmatrix} = 0$, then using properties of determinants, prove that at least one of the following statements is true: (a) p, q, r are in G. P., (b) α is a root of the equation $px^2 + 2qx + r = 0$.

26. Using integration, find the area of the region bounded by the curves $y = \sqrt{5 - x^2}$ and $y = |x - 1|$.

27. Evaluate: $\displaystyle\int_0^{\pi/2} \frac{x \sin x \, \cos x}{\sin^4 x + \cos^4 x} \, dx.$

OR

Evaluate: $\displaystyle\int_0^4 \left(x + e^{2x}\right) dx$ as the limit of a sum.

28. Find the equation of the plane through the point $(4, -3, 2)$ and perpendicular to the line of intersection of the planes $x - y + 2z - 3 = 0$ and $2x - y - 3z = 0$. Find the point of intersection of the line $\vec{r} = \hat{i} + 2\hat{j} - \hat{k} + \lambda(\hat{i} + 3\hat{j} - 9\hat{k})$ and the plane obtained above.

29. In a mid-day meal programme, an NGO wants to provide vitamin rich diet to the students of an MCD school. The dietician of the NGO wishes to mix two types of food in such a way that vitamin contents of the mixture contains at least 8 units of vitamin A and 10 units of vitamin C. Food 1 contains 2 units per kg of vitamin A and 1 unit per kg of vitamin C. Food 2 contains 1 unit per kg of vitamin A and 2 units per kg of vitamin C. It costs ₹ 50 per kg to purchase Food 1 and ₹ 70 per kg to purchase Food 2. Formulate the problem as LPP and solve it graphically for the minimum cost of such a mixture?

❑❑❑

Solution to Latest CBSE Sample Paper 2016-17

1. Given $\mathbb{R}$ = Set of all real numbers and $R = \{(a,b) : a \leq b^2;\ a, b \in \mathbb{R}\}$.
Reflexive: Since, $a \leq a^2$, is not true, for some $a \in \mathbb{R}$.
For $a = \frac{1}{2}$, we have $(a,a) \notin R$.
So, R is not reflexive. ■

2. Given that A is a matrix of order 3×3 and $|2A| = k|A|$.

We have, $$|2A| = k|A|$$
$$\Rightarrow \quad 2^3|A| = k|A| \qquad [\because |\alpha A| = \alpha^n A,\ \text{where } A \text{ is } n \times n \text{ matrix}]$$
$$\Rightarrow \quad 8|A| = k|A|$$
$$\Rightarrow \quad k = 8. \qquad [\because |A| \neq 0]$$ ■

3. Given two non-zero vectors $\vec{a}$ and $\vec{b}$ such that $|\vec{a} \times \vec{b}| = \vec{a} \cdot \vec{b}$.
Let θ be the angle between $\vec{a}$ and $\vec{b}$.

Now, $$|\vec{a} \times \vec{b}| = \vec{a} \cdot \vec{b}$$
$$\Rightarrow \quad |\vec{a}|\ |\vec{b}|\ \sin\theta = |\vec{a}|\ |\vec{b}|\ \cos\theta$$
$$\Rightarrow \quad \sin\theta = \cos\theta \qquad [\because |\vec{a}| \neq 0,\ |\vec{b}| \neq 0]$$
$$\Rightarrow \quad \tan\theta = 1$$
$$\Rightarrow \quad \theta = 45^\circ$$

Hence, angle between $\vec{a}$ and $\vec{b}$ is 45°. ■

4. Given that $*$ is a binary operation on $\mathbb{R}$ defined by $a * b = a + b - 2$, for all $a, b \in \mathbb{R}$.
Existence of Identity: Let $e \in \mathbb{R}$ be the identity element, then for every $a \in \mathbb{R}$
$$a * e = a \quad \text{and} \quad e * a = a.$$
$$\Rightarrow \quad a + e - 2 = a \quad \text{and} \quad e + a - 2 = a.$$
$$\Rightarrow \quad e = 2 \quad \text{and} \quad e = 2.$$

So, $e = 2 \in \mathbb{R}$.
Hence, $*$ has identity $e = 2$ on $\mathbb{R}$. ■

5. Let $y = \cot^{-1}\left(\frac{1}{\sqrt{x^2 - 1}}\right)$, where $x < -1$.
Putting $x = \sec\theta$, we get
$$y = \cot^{-1}\left(\frac{1}{\sqrt{\sec^2\theta - 1}}\right) = \cot^{-1}\left(\frac{1}{\tan\theta}\right) = \cot^{-1}(\cot\theta) = \theta = \sec^{-1}x.$$
Hence, $\sec^{-1}x$ is the required simplest form. ■

6. Let $A = [a_{ij}]$ be a skew symmetric matrix.
$$\therefore \quad A^T = -A$$

$$\Rightarrow \quad [a_{ij}]^T = -[a_{ij}]$$

$$\Rightarrow \quad [a_{ji}] = -[a_{ij}]$$

$$\Rightarrow \quad a_{ji} = -a_{ij} \qquad \text{for all possible values of } i, j$$

$$\Rightarrow \quad a_{ii} = -a_{ii} \qquad \text{for all possible values of } i$$

$$\Rightarrow \quad 2a_{ii} = 0 \qquad \text{for all possible values of } i$$

$$\Rightarrow \quad a_{ii} = 0 \qquad \text{for all possible values of } i$$

Hence, all the diagonal elements of A are zero. ■

7. Let $y = \tan^{-1}\left[\dfrac{5x}{1-6x^2}\right] = \tan^{-1}\left[\dfrac{2x+3x}{1-(2x)(3x)}\right] = \tan^{-1}(2x) + \tan^{-1}(3x)$.

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[\tan^{-1}(2x)\right] + \frac{d}{dx}\left[\tan^{-1}(3x)\right] = \frac{\frac{d}{dx}(2x)}{1+(2x)^2} + \frac{\frac{d}{dx}(3x)}{1+(3x)^2}$$

$$= \frac{2}{1+4x^2} + \frac{3}{1+9x^2}.$$ ■

8. Let $y = \log_e x$.

Then, $\dfrac{dy}{dx} = \dfrac{1}{x}$.

Choose $x = 4$ and $x + \Delta x = 4.01$

Then, (i) $\Delta x = 4.01 - x = 4.01 - 4 = 0.01$

(ii) $\dfrac{dy}{dx} = \dfrac{1}{x} = \dfrac{1}{4} = 0.25$

Now, $\Delta y \approx \left(\dfrac{dy}{dx}\right)\Delta x = (0.25)(0.01) = 0.0025$

Hence, required approximate change in $\log_e x$ is 0.0025 ■

9.
$$\int \left(\frac{1-x}{1+x^2}\right)^2 e^x\,dx = \int \frac{e^x\left[1+x^2-2x\right]}{(1+x^2)^2}\,dx$$

$$= \int e^x\left[\frac{1}{1+x^2} + \frac{(-2x)}{(1+x^2)^2}\right]dx$$

$$= \int \underbrace{e^x}_{\text{II}}\,\underbrace{\frac{1}{1+x^2}}_{\text{I}}\,dx + \int e^x\,\frac{(-2x)}{(1+x^2)^2}\,dx$$

$$= \left(\frac{1}{1+x^2}\right)e^x - \int \frac{(-2x)}{(1+x^2)^2}\,e^x\,dx + \int e^x\,\frac{(-2x)}{(1+x^2)^2}\,dx$$

$$= \frac{e^x}{1+x^2} + C.$$ ■

10. A circle which passes through the points $(a,0)$ and $(-a,0)$ has its centre on y-axis.

Let $(0,b)$ be the centre of the circle.

Then, radius of circle = Distance between $(0,b)$ and $(a,0) = \sqrt{a^2+b^2}$.

So, the equation of the family of circles which pass through the points $(a, 0)$ and $(-a, 0)$ is

$$x^2 + (y-b)^2 = a^2 + b^2, \text{ where } b \text{ is an arbitrary constant.}$$

$$\Rightarrow \quad x^2 + y^2 - 2by + b^2 = a^2 + b^2$$

$$\Rightarrow \quad x^2 + y^2 - a^2 = 2by$$

$$\Rightarrow \quad \frac{x^2 + y^2 - a^2}{y} = 2b$$

Differentiating both sides w.r.t. x, we get

$$\frac{y\left(2x + 2y\,\dfrac{dy}{dx}\right) - \left(x^2 + y^2 - a^2\right)\dfrac{dy}{dx}}{y^2} = 0$$

$$\Rightarrow \quad 2xy + 2y^2\,\frac{dy}{dx} - x^2\,\frac{dy}{dx} - y^2\,\frac{dy}{dx} + a^2\,\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \left(y^2 - x^2 + a^2\right)\frac{dy}{dx} + 2xy = 0,$$

which is the required differential equation. ■

11. Given $|\vec{a} + \vec{b}| = 60$, $|\vec{a} - \vec{b}| = 40$ and $|\vec{a}| = 22$.

Now,
$$\begin{aligned}
|\vec{a} + \vec{b}|^2 &= (\vec{a} + \vec{b})^2 \\
&= (\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) \\
&= \vec{a} \cdot (\vec{a} + \vec{b}) + \vec{b} \cdot (\vec{a} + \vec{b}) \\
&= \vec{a} \cdot \vec{a} + \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{a} + \vec{b} \cdot \vec{b} \\
&= |\vec{a}|^2 + \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{b} + |\vec{b}|^2 \\
\Rightarrow \quad |\vec{a} + \vec{b}|^2 &= |\vec{a}|^2 + 2\,\vec{a} \cdot \vec{b} + |\vec{b}|^2 \\
\Rightarrow \quad (60)^2 &= (22)^2 + 2\,\vec{a} \cdot \vec{b} + |\vec{b}|^2 \\
\Rightarrow \quad 3600 &= 484 + 2\,\vec{a} \cdot \vec{b} + |\vec{b}|^2 \\
\Rightarrow \quad 3116 &= 2\,\vec{a} \cdot \vec{b} + |\vec{b}|^2 \qquad \text{...(1)}
\end{aligned}$$

Now,
$$\begin{aligned}
|\vec{a} - \vec{b}|^2 &= (\vec{a} - \vec{b})^2 \\
&= (\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b}) \\
&= \vec{a} \cdot (\vec{a} - \vec{b}) - \vec{b} \cdot (\vec{a} - \vec{b}) \\
&= \vec{a} \cdot \vec{a} - \vec{a} \cdot \vec{b} - \vec{b} \cdot \vec{a} + \vec{b} \cdot \vec{b} \\
&= |\vec{a}|^2 - \vec{a} \cdot \vec{b} - \vec{a} \cdot \vec{b} + |\vec{b}|^2 \\
\Rightarrow \quad |\vec{a} - \vec{b}|^2 &= |\vec{a}|^2 - 2\,\vec{a} \cdot \vec{b} + |\vec{b}|^2 \\
\Rightarrow \quad (40)^2 &= (22)^2 - 2\,\vec{a} \cdot \vec{b} + |\vec{b}|^2 \\
\Rightarrow \quad 1600 &= 484 - 2\,\vec{a} \cdot \vec{b} + |\vec{b}|^2 \\
\Rightarrow \quad 1116 &= -2\,\vec{a} \cdot \vec{b} + |\vec{b}|^2 \qquad \text{...(2)}
\end{aligned}$$

On adding (1) and (2), we get

$$4232 = 2|\vec{b}|^2$$

$$\Rightarrow \quad |\vec{b}|^2 = 2116$$

$$\Rightarrow \quad |\vec{b}| = 46.$$

■

12. Given $P(A) = \dfrac{2}{5}$, $P(B) = \dfrac{1}{3}$ and $P(A \cap B) = \dfrac{1}{5}$.

Then,
$$P(\overline{A}|\overline{B}) = \frac{P(\overline{A} \cap \overline{B})}{P(\overline{B})} = \frac{P(\overline{A \cup B})}{P(\overline{B})} = \frac{1 - P(A \cup B)}{1 - P(B)}$$

$$= \frac{1 - [P(A) + P(B) - P(A \cap B)]}{1 - P(B)}$$

$$= \frac{1 - \dfrac{2}{5} - \dfrac{1}{3} + \dfrac{1}{5}}{1 - \left(\dfrac{1}{3}\right)} = \frac{7}{10}.$$

■

13. Given $A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$. Then, $|A| = \begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} = 1 + 4 = 5 \neq 0$.

So, A is non-singular and hence invertible.

Let A_{ij} denote cofactor of a_{ij} in $A = [a_{ij}]$, then

$A_{11} = (-1)^{1+1}(1) = 1,$ $\quad A_{21} = (-1)^{2+1}(-2) = 2,$

$A_{12} = (-1)^{1+2}(2) = -2,$ $\quad A_{22} = (-1)^{2+2}(1) = 1.$

$\therefore \text{adj } A = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}.$

and $A^{-1} = \dfrac{1}{|A|}(\text{adj } A) = \dfrac{1}{5}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}.$

Now, given system of equations is

$$x - 2y = -1$$
$$2x + y = 2$$

which can be written in matrix form as

$$\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

i.e.,
$$AX = B,$$

where $A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $B = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

Since, $|A| = 5 \neq 0$.

$\therefore$ Given system of equations is consistent and has unique solution given by $X = A^{-1}B$.

$$\therefore \quad \begin{bmatrix} x \\ y \end{bmatrix} = X = A^{-1}B = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}.$$

$\Rightarrow \quad x = \dfrac{3}{5}, y = \dfrac{4}{5}$

Hence, the required solution is $x = \dfrac{3}{5}, y = \dfrac{4}{5}$. ■

14. Given $f(x) = \begin{cases} 2x - 1 & \text{if } x < \dfrac{1}{2} \\ 3 - 6x & \text{if } x \geq \dfrac{1}{2} \end{cases}$.

Differentiability at $x = \dfrac{1}{2}$

$$Rf'\left(\frac{1}{2}\right) = \lim_{h \to 0}\left[\frac{f\left(\frac{1}{2} + h\right) - f\left(\frac{1}{2}\right)}{h}\right]$$

$$Lf'\left(\frac{1}{2}\right) = \lim_{h \to 0}\left[\frac{f\left(\frac{1}{2} - h\right) - f\left(\frac{1}{2}\right)}{-h}\right]$$

$$= \lim_{h\to 0}\left[\frac{\left\{3-6\left(\frac{1}{2}+h\right)\right\}-0}{h}\right]$$

$$= \lim_{h\to 0}\left[\frac{-6h}{h}\right] = \lim_{h\to 0}[-6] = -6.$$

$$= \lim_{h\to 0}\left[\frac{\left\{2\left(\frac{1}{2}-h\right)-1\right\}-0}{-h}\right]$$

$$= \lim_{h\to 0}\left[\frac{-2h}{-h}\right] = \lim_{h\to 0}[2] = 2.$$

Since, $Rf'\left(\frac{1}{2}\right) \neq Lf'\left(\frac{1}{2}\right)$.

Hence, f is not differentiable at $x = \frac{1}{2}$.

OR

Given $f(x) = \begin{cases} \dfrac{\sqrt{3}\ \sin x + \cos x}{x+\frac{\pi}{6}} & \text{if } x \neq -\frac{\pi}{6} \\ k & \text{if } x = -\frac{\pi}{6} \end{cases}$.

Since, f is continuous at $x = -\frac{\pi}{6}$.

$$\therefore \lim_{x\to(-\pi/6)+}[f(x)] = \lim_{x\to(-\pi/6)-}[f(x)] = f\left(-\frac{\pi}{6}\right).$$

$$\Rightarrow \quad f\left(-\frac{\pi}{6}\right) = \lim_{x\to(-\pi/6)+}[f(x)]$$

$$\Rightarrow \quad k = \lim_{x\to(-\pi/6)+}\left[\frac{\sqrt{3}\ \sin x + \cos x}{x+\frac{\pi}{6}}\right]$$

$$= \lim_{x\to(-\pi/6)+}\left[\frac{2\left\{\frac{\sqrt{3}}{2}\ \sin x + \frac{1}{2}\cos x\right\}}{x+\frac{\pi}{6}}\right]$$

$$= 2\lim_{x\to(-\pi/6)+}\left[\frac{\cos\frac{\pi}{6}\ \sin x + \sin\frac{\pi}{6}\cos x}{x+\frac{\pi}{6}}\right]$$

$$= 2\lim_{x\to(-\pi/6)+}\left[\frac{\sin\left\{x+\frac{\pi}{6}\right\}}{x+\frac{\pi}{6}}\right]$$

$$= 2\lim_{h\to 0}\left[\frac{\sin\left\{\left(-\frac{\pi}{6}+h\right)+\frac{\pi}{6}\right\}}{\left(-\frac{\pi}{6}+h\right)+\frac{\pi}{6}}\right] \qquad \left[\text{By putting } x = -\frac{\pi}{6}+h\right]$$

$$= 2\lim_{h\to 0}\left[\frac{\sin h}{h}\right]$$

$$= 2(1) = 2.$$

■

15. Given $x = a \sin pt$ and $y = b \cos pt$ ...(1)

Differentiating both sides w.r.t. t, we get

$$\frac{dx}{dt} = ap \cos pt$$

Differentiating both sides w.r.t. t, we get

$$\frac{dy}{dt} = -bp \sin pt$$

Then, $\displaystyle \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{-bp \sin pt}{ap \cos pt} = -\frac{b}{a} \tan pt.$

Differentiating both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2} = -\frac{b}{a} \frac{d}{dx} [\tan pt] \frac{1}{\left(\frac{dx}{dt}\right)}$$

$$= -\frac{b}{a} (p \sec^2 pt) \frac{1}{(ap \cos pt)}$$

$$= -\frac{b}{a^2 \cos^2 pt \, \cos pt}$$

$$= -\frac{b}{a^2(1 - \sin^2 pt) \, \cos pt}$$

$$= -\frac{b^2}{(a^2 - a^2 \sin^2 pt)(b \cos pt)}$$

$\Rightarrow$ $$\frac{d^2y}{dx^2} = -\frac{b^2}{(a^2 - x^2)y}$$ [Using (1)]

$\Rightarrow$ $$(a^2 - x^2)y \frac{d^2y}{dx^2} = -b^2$$

$\Rightarrow$ $$(a^2 - x^2)y \frac{d^2y}{dx^2} + b^2 = 0.$$ ■

16. Given curve is $2y = x^2$. ...(1)

Let the point of contact of required normal with the given curve be (α, β).

$\therefore$ $$2\beta = \alpha^2$$ ...(2)

Differentiating both sides of (1) w.r.t. x, we get

$$2 \frac{dy}{dx} = 2x$$

$\Rightarrow$ $$\frac{dy}{dx} = x$$

$\Rightarrow$ $$\left(\frac{dy}{dx}\right)_{(\alpha,\beta)} = \alpha$$

Now, equation of normal at (α, β) is $\displaystyle y - \beta = -\frac{1}{\left(\frac{dy}{dx}\right)_{(\alpha,\beta)}}(x - \alpha)$

i.e., $$y - \beta = -\frac{1}{\alpha}(x - \alpha) \qquad ...(3)$$

Given that normal line passes through the point $(2, 1)$.

$$\therefore \qquad 1 - \beta = -\frac{1}{\alpha}(2 - \alpha)$$

$$\Rightarrow \qquad 1 - \frac{\alpha^2}{2} = -\frac{1}{\alpha}(2 - \alpha) \qquad \text{[Using (2)]}$$

$$\Rightarrow \qquad 2\alpha - \alpha^3 = -4 + 2\alpha$$

$$\Rightarrow \qquad \alpha^3 = 4$$

$$\Rightarrow \qquad \alpha = 4^{1/3} = 2^{2/3}$$

Putting $\alpha = 2^{2/3}$ in (2), we get $\beta = \dfrac{2^{4/3}}{2} = 2^{1/3}$.

Putting $\alpha = 2^{2/3}$ and $\beta = 2^{1/3}$ in (3), we get

Equation of normal at $\left(2^{2/3}, 2^{1/3}\right)$ is

$$y - 2^{1/3} = -\frac{1}{2^{2/3}}(x - 2^{2/3})$$

i.e., $$2^{2/3}y - 2 = -x + 2^{2/3}$$

i.e., $$x + 2^{2/3}y = 2 + 2^{2/3}.$$

OR

Given $f(x) = \sin^4 x + \cos^4 x$ on $\left[0, \dfrac{\pi}{2}\right]$.

Then, $f'(x) = 4\sin^3 x \, \cos x - 4\cos^3 x \, \sin x$

$= -4\sin x \, \cos x(\cos^2 x - \sin^2 x)$

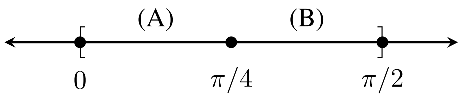
$= -2(\sin 2x)(\cos 2x) = -\sin 4x.$

Now, $f'(x) = 0 \quad \Rightarrow \quad -\sin 4x = 0$

$\Rightarrow \quad \sin 4x = 0$

$\Rightarrow \quad 4x = 0, \pi, 2\pi$

$\Rightarrow \quad x = 0, \dfrac{\pi}{4}, \dfrac{\pi}{2} \in \left[0, \dfrac{\pi}{2}\right].$



	Sub-interval	Test Point	Test Value	Sign of $f'(x)$	Conclusion
(A)	$\left(0, \frac{\pi}{4}\right)$	$\frac{\pi}{6}$	$f'\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}$	$(-)$	$f'(x) < 0$
(B)	$\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$	$\frac{\pi}{3}$	$f'\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$	$(+)$	$f'(x) > 0$

Hence, $f(x)$ is strictly increasing on $\left(\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$ and strictly decreasing on $\left(0, \dfrac{\pi}{4}\right)$. ■

17. Suppose that the magazine seller increases the annual subscription by ₹ x per subscriber.

Then, the number of subscribers who will discontinue the services are x.

Now, total revenue of the company after the increment is given by

$$R = \text{ Number of subscribers left } \times \text{ New subscription charges per subscriber}$$
$$= (500 - x)(300 + x)$$
$$= -x^2 + 200x + 150000.$$

Then, $\dfrac{dR}{dx} = -2x + 200.$

Now, $\dfrac{dR}{dx} = 0 \quad \Rightarrow \quad -2x + 200 = 0$

$\Rightarrow \quad x = 100$

Also, $\dfrac{d^2R}{dx^2} = -2$

$\Rightarrow \quad \left(\dfrac{d^2R}{dx^2}\right)_{x=100} = -2 < 0$

So, by second derivative test, R is maximum when $x = 100$.

Hence, required increase in the subscription charges per subscriber is ₹ 100.

Magazine supply us with a variety of news and contribute in the development of our knowledge. ■

18. Let $I = \displaystyle\int \frac{\sin x}{(\cos^2 x + 1)(\cos^2 x + 4)}\, dx$

Put $\cos x = y$

$\Rightarrow \quad \sin x\, dx = -dy$

$\therefore\ I = -\displaystyle\int \frac{1}{(y^2 + 1)(y^2 + 4)}\, dy$ ...(1)

Putting $y^2 = t$ in the fraction $\dfrac{1}{(y^2 + 1)\,(y^2 + 4)}$, we get

$$\frac{1}{(y^2 + 1)\,(y^2 + 4)} = \frac{1}{(t + 1)(t + 4)}$$

Put $\dfrac{1}{(t + 1)(t + 4)} = \dfrac{A}{t + 1} + \dfrac{B}{t + 4}$ ...(2)

$\Rightarrow \quad 1 = A(t + 4) + B(t + 1)$

$\Rightarrow \quad 1 = At + 4A + Bt + B$

On comparing coefficients of t and constant terms on both sides, we get

$$0 = A + B, \qquad 1 = 4A + B$$

On solving above equations, we get

$$A = \frac{1}{3}, \qquad B = -\frac{1}{3}$$

Putting values of A and B in (2), we get

$$\frac{1}{(t+1)(t+4)} = \frac{1}{3(t+1)} - \frac{1}{3(t+4)}$$

$$\therefore \quad \frac{1}{(y^2+1)(y^2+4)} = \frac{1}{3(y^2+1)} - \frac{1}{3(y^2+4)} \qquad ...(3)$$

From (1) and (3), we get

$$I = -\frac{1}{3}\int \frac{1}{y^2+1}\,dy + \frac{1}{3}\int \frac{1}{y^2+4}\,dy$$

$$= -\frac{1}{3}\tan^{-1}y + \frac{1}{6}\tan^{-1}\left(\frac{y}{2}\right) + C$$

$$= -\frac{1}{3}\tan^{-1}(\cos x) + \frac{1}{6}\tan^{-1}\left(\frac{\cos x}{2}\right) + C.$$

■

19. Given differential equation is

$$(1+\tan y)(dx - dy) + 2x\,dy = 0$$

$$\Rightarrow \quad (1+\tan y)\,dx = (1+\tan y - 2x)\,dy$$

$$\Rightarrow \quad \frac{dx}{dy} = 1 - \frac{2x}{1+\tan y}$$

$$\Rightarrow \quad \frac{dx}{dy} + \frac{2}{1+\tan y}\,x = 1,$$

which is of the form $\dfrac{dx}{dy} + Px = Q$ with $P = \dfrac{2}{1+\tan y}$ and $Q = 1$.

Now, integrating factor,

$$I.F. = e^{\int P\,dy} = e^{\int \frac{2}{1+\tan y}\,dy} = e^{\int \frac{2\cos y}{\cos y+\sin y}\,dy} = e^{\int \frac{(\cos y+\sin y)+(\cos y-\sin y)}{\cos y+\sin y}\,dy}$$

$$= e^{\int \left(1+\frac{\cos y-\sin y}{\sin y+\cos y}\right)dy}$$

$$= e^{y+\log(\sin y+\cos y)}$$

$$= e^{y}(\sin y+\cos y)$$

$\therefore$ General solution of given differential equation is

$$x(I.F.) = \int Q(I.F.)\,dy + C$$

$$\Rightarrow \quad x\,e^y(\sin y+\cos y) = \int e^y(\sin y+\cos y)\,dy + C$$

$$\Rightarrow \quad x\,e^y(\sin y+\cos y) = \int \underbrace{e^y}_{\text{II}}\,\underbrace{\sin y}_{\text{I}}\,dy + \int e^y\cos y\,dy + C$$

$$\Rightarrow \quad x\,e^y(\sin y+\cos y) = \sin y\int e^y\,dy - \int\left[\frac{d}{dy}\{\sin y\}\int e^y\,dy\right]dy + \int e^y\cos y\,dy + C$$

$$\Rightarrow \quad x\,e^y(\sin y+\cos y) = e^y\sin y - \int[(\cos y)(e^y)]\,dy + \int e^y\cos y\,dy + C$$

$$\Rightarrow \quad x\,e^y(\sin y+\cos y) = e^y\sin y + C,$$

which is the required solution.

OR

Given differential equation is

$$(1 + e^{x/y})\ dx + e^{x/y}\left(1 - \frac{x}{y}\right) dy = 0$$

$$\Rightarrow \qquad (1 + e^{x/y})\ dx = e^{x/y}\left(\frac{x}{y} - 1\right) dy$$

$$\Rightarrow \qquad \frac{dx}{dy} = \frac{e^{x/y}\left(\frac{x}{y} - 1\right)}{1 + e^{x/y}} \qquad ...(1)$$

Putting $x = vy$ and $\frac{dx}{dy} = v + y\ \frac{dv}{dy}$ in (1), we get

$$v + y\ \frac{dv}{dy} = \frac{e^v(v-1)}{1+e^v}$$

$$\Rightarrow \qquad y\ \frac{dv}{dy} = \frac{e^v(v-1)}{1+e^v} - v$$

$$\Rightarrow \qquad y\ \frac{dv}{dy} = \frac{-e^v - v}{1+e^v}$$

$$\Rightarrow \qquad \frac{1+e^v}{v+e^v}\ dv = -\frac{1}{y}\ dy$$

Integrating both sides, we get

$$\int \frac{1+e^v}{v+e^v}\ dv = -\int \frac{1}{y}\ dy$$

$$\Rightarrow \qquad \log|v + e^v| = -\log|y| + C$$

$$\Rightarrow \qquad \log\left|\frac{x}{y} + e^{x/y}\right| = -\log|y| + C,$$

which is the required solution. ■

20. L.H.S. $= \vec{a} \cdot \{(\vec{b} + \vec{c}) \times (\vec{a} + 2\vec{b} + 3\vec{c})\}$

$$= \vec{a} \cdot \{(\vec{b} \times \vec{a}) + 2(\vec{b} \times \vec{b}) + 3(\vec{b} \times \vec{c}) + (\vec{c} \times \vec{a}) + 2(\vec{c} \times \vec{b}) + 3(\vec{c} \times \vec{c})\}$$

$$= \vec{a} \cdot \{(\vec{b} \times \vec{a}) + 3(\vec{b} \times \vec{c}) + (\vec{c} \times \vec{a}) + 2(\vec{c} \times \vec{b})\} \qquad \left[\begin{array}{r}\because \vec{b} \times \vec{b} = \vec{0} \\ \vec{c} \times \vec{c} = \vec{0}\end{array}\right]$$

$$= \vec{a} \cdot (\vec{b} \times \vec{a}) + 3\vec{a} \cdot (\vec{b} \times \vec{c}) + \vec{a} \cdot (\vec{c} \times \vec{a}) + 2\vec{a} \cdot (\vec{c} \times \vec{b})$$

$$= 0 + 3\vec{a} \cdot (\vec{b} \times \vec{c}) + 0 + 2\vec{a} \cdot (\vec{c} \times \vec{b})$$

$$= 3\vec{a} \cdot (\vec{b} \times \vec{c}) - 2\vec{a} \cdot (\vec{b} \times \vec{c}) \qquad [\because \vec{c} \times \vec{b} = -\vec{b} \times \vec{c}]$$

$$= \vec{a} \cdot (\vec{b} \times \vec{c})$$

$$= [\,\vec{a}\ \ \vec{b}\ \ \vec{c}\,] = \text{ R.H.S.}$$ ■

21. The Cartesian equations of given lines are

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-a}{4} \text{ and } \frac{x-4}{5} = \frac{y-1}{2} = z$$

$\therefore$ The vector equations of given lines are

$$\vec{r} = (\hat{i} + 2\hat{j} + a\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 4\hat{k}) \quad \text{and} \quad \vec{r} = (4\hat{i} + \hat{j} + 0\hat{k}) + \mu(5\hat{i} + 2\hat{j} + \hat{k})$$

$$= \vec{a}_1 + \lambda\vec{b}_1 \qquad\qquad = \vec{a}_2 + \mu\vec{b}_2$$

where $\vec{a}_1 = \hat{i} + 2\hat{j} + a\hat{k}$, $\vec{b}_1 = 2\hat{i} + 3\hat{j} + 4\hat{k}$,

$\vec{a}_2 = 4\hat{i} + \hat{j} + 0\hat{k}$, $\vec{b}_2 = 5\hat{i} + 2\hat{j} + \hat{k}$.

Here, $\vec{b_1} \neq \vec{b_2}$.

$\therefore$ Given lines are either intersecting or skew lines.

It is given that the given lines are skew lines.

So, the given lines are non-intersecting.

Equation of first line is

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-a}{4}.$$

Let $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-a}{4} = \alpha$.

Then, $x = 2\alpha + 1$, $y = 3\alpha + 2$, $z = 4\alpha + a$.

$\therefore$ General point on first line is

$$(2\alpha + 1, 3\alpha + 2, 4\alpha + a) \qquad ...(1)$$

Equation of second line is

$$\frac{x-4}{5} = \frac{y-1}{2} = z.$$

Let $\frac{x-4}{5} = \frac{y-1}{2} = z = \beta$.

Then, $x = 5\beta + 4$, $y = 2\beta + 1$, $z = \beta$.

$\therefore$ General point on second line is

$$(5\beta + 4, 2\beta + 1, \beta) \qquad ...(2)$$

On equating the coordinates of (1) and (2), we get

$$2\alpha + 1 = 5\beta + 4 \quad \Rightarrow \quad 2\alpha - 5\beta = 3 \qquad ...(3)$$

$$3\alpha + 2 = 2\beta + 1 \quad \Rightarrow \quad 3\alpha - 2\beta = -1 \qquad ...(4)$$

$$4\alpha + a = \beta \quad \Rightarrow \quad 4\alpha - \beta = -a \qquad ...(5)$$

From (3) and (4), we get $\alpha = -1$, $\beta = -1$.

Putting $\alpha = -1$, $\beta = -1$ in (5), we get $a = 3$.

So, the given lines intersect if $a = 3$ and the given lines do not intersect if $a \neq 3$.

Hence, the given lines are skew if $a \neq 3$. ■

22. Given Bag : 4 green and 6 white balls

Let B_1, B_2 and E be the events defined as follows:

B_1 : First ball is green, B_2 : First ball is white and E : Second ball is white.

Then, $P(B_1) = \frac{4}{10}$, $P(E|B_1) = \frac{6}{9}$,

$P(B_2) = \frac{6}{10}$, $P(E|B_2) = \frac{5}{9}$.

By Bayes' Theorem, we have

$$P(B_2|E) = \frac{P(B_2)P(E|B_2)}{P(B_1)P(E|B_1) + P(B_2)P(E|B_2)} = \frac{\left(\frac{6}{10}\right)\left(\frac{5}{9}\right)}{\left(\frac{4}{10}\right)\left(\frac{6}{9}\right) + \left(\frac{6}{10}\right)\left(\frac{5}{9}\right)} = \frac{5}{9}.$$

Hence, required probability $= \frac{5}{9}$. ■

23. Total number of cards $= 52$.

Number of cards drawn $= 2$ (with replacement).

Suppose that 'number of diamond cards' is considered as success.

Let X be a random variable defined as the number of successes.

Then, X can attain the values 0, 1, 2.

Now, $P(X = 0) = P(\text{both are non-diamond cards})$ [Fix Case]

$$= \frac{39}{52} \times \frac{39}{52} = \frac{9}{16}.$$

$P(X = 1) = P(\text{one is diamond card and one is non-diamond card})$ [Unfix Case]

$$= P(\text{first is diamond card and next is non-diamond card}) \times \frac{2!}{1!\,1!}$$

$$= \frac{13}{52} \times \frac{39}{52} \times \frac{2!}{1!\,1!} = \frac{6}{16}.$$

$P(X = 2) = P(\text{both are diamond cards})$ [Fix Case]

$$= \frac{13}{52} \times \frac{13}{52} = \frac{1}{16}.$$

So, probability distribution of X is given by

X	0	1	2
$P(X)$	$\frac{9}{16}$	$\frac{6}{16}$	$\frac{1}{16}$

Calculation of Mean and Variance

X	$P(X)$	$X\ P(X)$	$X^2\ P(X)$
0	$\frac{9}{16}$	0	0
1	$\frac{6}{16}$	$\frac{6}{16}$	$\frac{6}{16}$
2	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{4}{16}$
		$\sum X\ P(X) = \frac{8}{16} = \frac{1}{2}$	$\sum X^2\ P(X) = \frac{10}{16} = \frac{5}{8}$

Hence, mean $= \overline{X} = \sum X\ P(X) = \frac{1}{2}$,

$$\text{variance} = var(X) = \sum X^2\ P(X) - \left[\sum X\ P(X)\right]^2 = \frac{5}{8} - \left(\frac{1}{2}\right)^2 = \frac{3}{8}.$$ ■

24. Given $f : [0, \infty) \to \mathbb{R}$ defined by $f(x) = 9x^2 + 6x - 5$.

(i) **One-one:** Let $x_1, x_2 \in [0, \infty)$ be any two elements.

Then, $f(x_1) = f(x_2)$

$\Rightarrow \quad 9x_1^2 + 6x_1 - 5 = 9x_2^2 + 6x_2 - 5$

$\Rightarrow \quad 9x_1^2 + 6x_1 = 9x_2^2 + 6x_2$

$\Rightarrow \quad 9(x_1^2 - x_2^2) + 6(x_1 - x_2) = 0$

$\Rightarrow \quad (x_1 - x_2)[9(x_1 + x_2) + 6] = 0$

$$\Rightarrow \quad x_1 - x_2 = 0 \quad [\because 9(x_1 + x_2) + 6 \neq 0]$$
$$\Rightarrow \quad x_1 = x_2$$

So, f is one-one.

(ii) **Onto:** Let $y \in \mathbb{R}$ be any element.

Then,
$$f(x) = y$$
$$\Rightarrow \quad 9x^2 + 6x - 5 = y$$
$$\Rightarrow \quad 9x^2 + 6x - 5 - y = 0$$
$$\Rightarrow \quad x = \frac{-6 \pm \sqrt{36 - 4(9)(-5-y)}}{18}$$
$$= \frac{-6 \pm \sqrt{36 + 36(5+y)}}{18}$$
$$= \frac{-6 \pm \sqrt{36(1+5+y)}}{18}$$
$$= \frac{-6 \pm 6\sqrt{6+y}}{18}$$
$$\Rightarrow \quad x = \frac{-1 \pm \sqrt{y+6}}{3}$$

For $y = -7 \in \mathbb{R}$, we have $x \notin [0, \infty)$.

So, f is not onto.

Hence, f is not a bijection and hence not invertible.

We know that f is onto if the co-domain of f equals its range.

Now, $f(x) = 9x^2 + 6x - 5 = (3x+1)^2 - 6$.

We have,
$$0 \leq x < \infty$$
$$\Rightarrow \quad 0 \leq 3x < \infty$$
$$\Rightarrow \quad 1 \leq 3x + 1 < \infty$$
$$\Rightarrow \quad 1 \leq (3x+1)^2 < \infty$$
$$\Rightarrow \quad -5 \leq (3x+1)^2 - 6 < \infty$$
$$\Rightarrow \quad -5 \leq f(x) < \infty$$

So, Range $(f) = [-5, \infty)$.

So, the modified function is $f : [0, \infty) \to [-5, \infty)$.

We know that every function is onto up to its range.

So, f is onto.

Thus, f is a bijection and hence invertible.

For every $y \in [-5, \infty)$, we have $x = \dfrac{\sqrt{y+6} - 1}{3} \in [0, \infty)$. ...(1)

Since, $f^{-1} : [-5, \infty) \to [0, \infty)$ exists and from (1), we have

$$f^{-1}(y) = \frac{\sqrt{y+6} - 1}{3}. \quad [\because f(x) = y \Leftrightarrow x = f^{-1}(y)]$$

Hence, inverse of f is given by $f^{-1}(x) = \dfrac{\sqrt{x+6} - 1}{3}$.

OR

Given binary operation $*$ on $\mathbb{Q} \times \mathbb{Q}$, defined by $(a, b) * (c, d) = (ac, b + ad)$.

Commutativity: Let $(a, b), (c, d) \in \mathbb{Q} \times \mathbb{Q}$ be any two elements.

Then, $(a, b) * (c, d) = (ac, b + ad)$ and $(c, d) * (a, b) = (ca, d + cb)$.

For $(a, b) = (0, 1)$, $(c, d) = (1, 1)$, we have $(a, b) * (c, d) = (0, 1)$ and $(c, d) * (a, b) = (0, 2)$.

$\Rightarrow (a, b) * (c, d) \neq (c, d) * (a, b)$.

Hence, $*$ is not commutative on $\mathbb{Q} \times \mathbb{Q}$.

Associativity: Let $(a, b), (c, d), (e, f) \in \mathbb{Q} \times \mathbb{Q}$ be any three elements.

Then, $\quad ((a, b) * (c, d)) * (e, f) = (ac, b + ad) * (e, f) = (ace, b + ad + acf)$

and $\quad (a, b) * ((c, d) * (e, f)) = (a, b) * (ce, d + cf) = (ace, b + ad + acf)$.

$\Rightarrow \quad ((a, b) * (c, d)) * (e, f) = (a, b) * ((c, d) * (e, f))$.

Hence, $*$ is associative on $\mathbb{Q} \times \mathbb{Q}$.

Existence of Identity: Let $(e_1, e_2) \in \mathbb{Q} \times \mathbb{Q}$ be the identity element, then for every $(a, b) \in \mathbb{Q} \times \mathbb{Q}$

$$(a, b) * (e_1, e_2) = (a, b) \quad \text{and} \quad (e_1, e_2) * (a, b) = (a, b).$$

$$\Rightarrow \quad (ae_1, b + ae_2) = (a, b) \quad \text{and} \quad (e_1a, e_2 + e_1b) = (a, b).$$

$$\Rightarrow \quad ae_1 = a, b + ae_2 = b \quad \text{and} \quad e_1a = a, e_2 + e_1b = b.$$

$$\Rightarrow \quad e_1 = 1, e_2 = 0 \quad \text{and} \quad e_1 = 1, e_2 = 0.$$

$\Rightarrow (e_1, e_2) = (1, 0) \in \mathbb{Q} \times \mathbb{Q}$.

Hence, $*$ has identity $(e_1, e_2) = (1, 0)$ on $\mathbb{Q} \times \mathbb{Q}$.

Existence of Inverse: Let $(a, b) \in \mathbb{Q} \times \mathbb{Q}$ be any invertible element and let $(c, d) \in \mathbb{Q} \times \mathbb{Q}$ be its inverse, then

$$(a, b) * (c, d) = (e_1, e_2) \quad \text{and} \quad (c, d) * (a, b) = (e_1, e_2).$$

$$\Rightarrow \quad (ac, b + ad) = (1, 0) \quad \text{and} \quad (ca, d + cb) = (1, 0).$$

$$\Rightarrow \quad ac = 1, b + ad = 0 \quad \text{and} \quad ca = 1, d + cb = 0.$$

$$\Rightarrow \quad c = \frac{1}{a}, d = -\frac{b}{a} \quad \text{and} \quad c = \frac{1}{a}, d = -cb.$$

$$\Rightarrow \quad c = \frac{1}{a}, d = -\frac{b}{a} \quad \text{and} \quad c = \frac{1}{a}, d = -\frac{b}{a}.$$

Since, $(c, d) \in \mathbb{Q} \times \mathbb{Q}$ for every $(a, b) \in \mathbb{Q} \times \mathbb{Q}$ with $a \neq 0$.

So, (a, b) with $a \neq 0$ are the invertible elements of $*$ in $\mathbb{Q} \times \mathbb{Q}$ and their inverses are $\left(\frac{1}{a}, -\frac{b}{a}\right)$. ■

25. L.H.S. $= \begin{vmatrix} \frac{(a+b)^2}{c} & c & c \\ a & \frac{(b+c)^2}{a} & a \\ b & b & \frac{(c+a)^2}{b} \end{vmatrix}$

$$= \frac{a^2b^2c^2}{a^2b^2c^2}\begin{vmatrix} \frac{(a+b)^2}{c} & c & c \\ a & \frac{(b+c)^2}{a} & a \\ b & b & \frac{(c+a)^2}{b} \end{vmatrix}$$

$$= \frac{1}{a^2b^2c^2}\begin{vmatrix} c(a+b)^2 & ca^2 & cb^2 \\ ac^2 & a(b+c)^2 & ab^2 \\ bc^2 & ba^2 & b(c+a)^2 \end{vmatrix} \quad \begin{bmatrix}\text{On multiplying } C_1 \text{ by } c^2, \\ C_2 \text{ by } a^2 \text{ and } C_3 \text{ by } b^2\end{bmatrix}$$

$$= \frac{abc}{a^2b^2c^2}\begin{vmatrix} (a+b)^2 & a^2 & b^2 \\ c^2 & (b+c)^2 & b^2 \\ c^2 & a^2 & (c+a)^2 \end{vmatrix} \quad \begin{bmatrix}\text{On taking common } c \text{ from } R_1, \\ a \text{ from } R_2 \text{ and } b \text{ from } R_3\end{bmatrix}$$

$$= \frac{1}{abc}\begin{vmatrix} (a+b)^2-c^2 & 0 & b^2-(c+a)^2 \\ 0 & (b+c)^2-a^2 & b^2-(c+a)^2 \\ c^2 & a^2 & (c+a)^2 \end{vmatrix} \quad \begin{bmatrix}\text{On applying} \\ R_1 \to R_1 - R_3 \\ R_2 \to R_2 - R_3\end{bmatrix}$$

$$= \frac{1}{abc}\begin{vmatrix} (a+b+c)(a+b-c) & 0 & (b+c+a)(b-c-a) \\ 0 & (b+c+a)(b+c-a) & (b+c+a)(b-c-a) \\ c^2 & a^2 & (c+a)^2 \end{vmatrix}$$

$$= \frac{(a+b+c)^2}{abc}\begin{vmatrix} a+b-c & 0 & b-c-a \\ 0 & b+c-a & b-c-a \\ c^2 & a^2 & (c+a)^2 \end{vmatrix} \quad \begin{bmatrix}\text{On taking common} \\ (a+b+c) \text{ from } R_1 \text{ and } R_2\end{bmatrix}$$

$$= \frac{(a+b+c)^2}{abc}\begin{vmatrix} a+b-c & 0 & -2a \\ 0 & b+c-a & -2c \\ c^2 & a^2 & 2ac \end{vmatrix} \quad \begin{bmatrix}\text{On applying} \\ C_3 \to C_3 - C_2 - C_1\end{bmatrix}$$

$$= \frac{(a+b+c)^2}{abc}\begin{vmatrix} a+b & \frac{a^2}{c} & 0 \\ \frac{c^2}{a} & b+c & 0 \\ c^2 & a^2 & 2ac \end{vmatrix} \quad \begin{bmatrix}\text{On applying} \\ R_1 \to R_1 + \frac{1}{c}R_3 \\ R_2 \to R_2 + \frac{1}{a}R_3\end{bmatrix}$$

$$= \frac{(a+b+c)^2}{abc}\left[0 - 0 + 2ac\{(a+b)(b+c) - ac\}\right] \quad \left[\text{On expanding along } C_3\right]$$

$$= \frac{(a+b+c)^2}{abc}\left[2ac\left\{ab + ac + b^2 + bc - ac\right\}\right]$$

$$= \frac{(a+b+c)^2}{abc}\left[2abc\{a+b+c\}\right] = 2(a+b+c)^3 = \text{R.H.S.}$$

OR

Let $$\Delta = \begin{vmatrix} p & q & p\alpha + q \\ q & r & q\alpha + r \\ p\alpha + q & q\alpha + r & 0 \end{vmatrix}$$

$$= \begin{vmatrix} p & q & 0 \\ q & r & 0 \\ p\alpha + q & q\alpha + r & -p\alpha^2 - 2q\alpha - r \end{vmatrix} \qquad \left[\begin{array}{c}\text{On applying} \\ C_3 \to C_3 - \alpha C_1 - C_2\end{array}\right]$$

$$= (-p\alpha^2 - 2q\alpha - r)\begin{vmatrix} p & q & 0 \\ q & r & 0 \\ p\alpha + q & q\alpha + r & 1 \end{vmatrix} \qquad \left[\begin{array}{c}\text{On taking common} \\ (-p\alpha^2 - 2q\alpha - r) \text{ from } C_3\end{array}\right]$$

$$= (-p\alpha^2 - 2q\alpha - r)[0 - 0 + 1\{pr - q^2\}] \qquad \left[\text{On expanding along } C_3\right]$$

$$\therefore \quad \Delta = (q^2 - pr)(p\alpha^2 + 2q\alpha + r)$$

Given that $\Delta = 0$

$$\Rightarrow \quad (q^2 - pr)(p\alpha^2 + 2q\alpha + r) = 0$$

$$\Rightarrow \quad \text{either } q^2 - pr = 0 \text{ or } p\alpha^2 + 2q\alpha + r = 0.$$

Hence, either p, q, r are in G. P. or α is a root of the equation $px^2 + 2qx + r = 0$. ■

26. Given curves are $y = \sqrt{5 - x^2}$ and $y = |x - 1| = \begin{cases} x - 1 & \text{if } x \geq 1 \\ -x + 1 & \text{if } x \leq 1 \end{cases}$.

$\therefore$ Given curves are

(i) $y = \sqrt{5 - x^2}$ (ii) $y = x - 1$, if $x \geq 1$ (iii) $y = -x + 1$, if $x \leq 1$

To find points of intersection

From (i) and (ii), we have

$$\sqrt{5 - x^2} = x - 1$$

$$\Rightarrow \quad 5 - x^2 = x^2 + 1 - 2x$$

$$\Rightarrow \quad x^2 - x - 2 = 0$$

$$\Rightarrow \quad (x - 2)(x + 1) = 0$$

$$\Rightarrow \quad x = 2 \qquad [\because x \geq 1]$$

$\therefore$ Curves (i) and (ii) intersect each other at $(2, 1)$.

From (i) and (iii), we have

$$\sqrt{5 - x^2} = -x + 1$$

$$\Rightarrow \quad 5 - x^2 = x^2 + 1 - 2x$$

$$\Rightarrow \quad x^2 - x - 2 = 0$$

$$\Rightarrow \quad (x - 2)(x + 1) = 0$$

$$\Rightarrow \quad x = -1 \qquad [\because x \leq 1]$$

$\therefore$ Curves (i) and (iii) intersect each other at $(-1, 2)$.

To plot the required region

$y = \sqrt{5 - x^2}$

x	0	$\sqrt{5}$	$-\sqrt{5}$	2	-1
y	$\sqrt{5}$	0	0	1	2

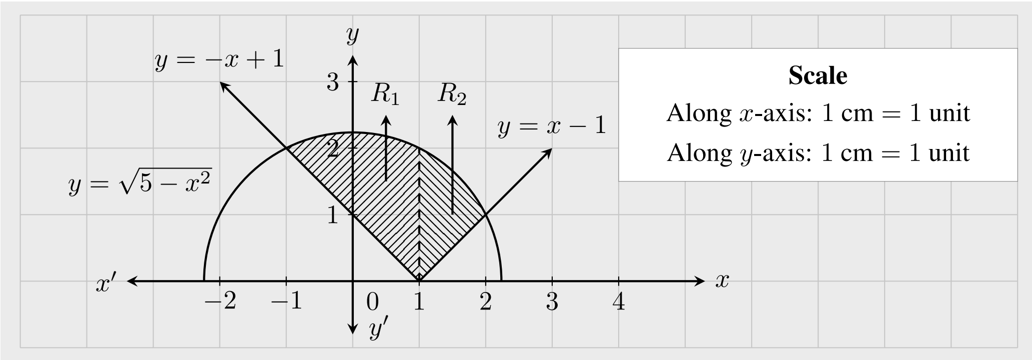
$y = x - 1$ (if $x \geq 1$)

x	0	2
y	-1	1

$y = -x + 1$ (if $x \leq 1$)

x	0	-1
y	1	2

The required region is shown shaded in the graph.



To find the area of required region

$A =$ Area of region R_1 + Area of region R_2

$$= \int_{-1}^{1} \left[\sqrt{5-x^2} - (-x+1)\right] dx + \int_{1}^{2} \left[\sqrt{5-x^2} - (x-1)\right] dx$$

$$= \left[\frac{x}{2}\sqrt{5-x^2} + \frac{5}{2}\sin^{-1}\frac{x}{\sqrt{5}} + \frac{x^2}{2} - x\right]_{-1}^{1} + \left[\frac{x}{2}\sqrt{5-x^2} + \frac{5}{2}\sin^{-1}\frac{x}{\sqrt{5}} - \frac{x^2}{2} + x\right]_{1}^{2}$$

$$= \left[\left(1 + \frac{5}{2}\sin^{-1}\frac{1}{\sqrt{5}} + \frac{1}{2} - 1\right) - \left(-1 - \frac{5}{2}\sin^{-1}\frac{1}{\sqrt{5}} + \frac{1}{2} + 1\right)\right]$$

$$+ \left[\left(1 + \frac{5}{2}\sin^{-1}\frac{2}{\sqrt{5}} - 2 + 2\right) - \left(1 + \frac{5}{2}\sin^{-1}\frac{1}{\sqrt{5}} - \frac{1}{2} + 1\right)\right]$$

$$= 5\sin^{-1}\frac{1}{\sqrt{5}} + \frac{5}{2}\sin^{-1}\frac{2}{\sqrt{5}} - \frac{5}{2}\sin^{-1}\frac{1}{\sqrt{5}} - \frac{1}{2}$$

$$= \frac{5}{2}\sin^{-1}\frac{1}{\sqrt{5}} + \frac{5}{2}\sin^{-1}\frac{2}{\sqrt{5}} - \frac{1}{2}.$$

Hence, required area $= \left[\frac{5}{2}\sin^{-1}\frac{1}{\sqrt{5}} + \frac{5}{2}\sin^{-1}\frac{2}{\sqrt{5}} - \frac{1}{2}\right]$ sq. units. ■

27. Let $$I = \int_0^{\pi/2} \frac{x\sin x\cos x}{\sin^4 x + \cos^4 x}\,dx \qquad ...(1)$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x\right)\sin\left(\frac{\pi}{2} - x\right)\cos\left(\frac{\pi}{2} - x\right)}{\sin^4\left(\frac{\pi}{2} - x\right) + \cos^4\left(\frac{\pi}{2} - x\right)}\,dx$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x\right)\cos x\sin x}{\cos^4 x + \sin^4 x}\,dx \qquad ...(2)$$

On adding (1) and (2), we get

$$2I = \int_0^{\pi/2} \frac{x\sin x\cos x}{\sin^4 x + \cos^4 x}\,dx + \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x\right)\cos x\sin x}{\cos^4 x + \sin^4 x}\,dx$$

$$\Rightarrow \quad 2I = \int_0^{\pi/2} \frac{\left(x + \frac{\pi}{2} - x\right)\sin x\cos x}{\sin^4 x + \cos^4 x}\,dx$$

$$\Rightarrow \qquad I = \frac{\pi}{4}\int_0^{\pi/2} \frac{\sin x\ \cos x}{\sin^4 x + \cos^4 x}\, dx$$

$$= \frac{\pi}{4}\int_0^{\pi/2} \frac{\tan x\ \sec^2 x}{\tan^4 x + 1}\, dx \qquad \left[\begin{array}{c}\text{On dividing numerator and}\\ \text{denominator by } \cos^4 x\end{array}\right]$$

Put $\tan^2 x = y$

$\Rightarrow \quad \tan x\ \sec^2 x\, dx = \frac{1}{2}\, dy$

If $x = 0$, then $y = 0$

If $x = \frac{\pi}{2}$, then $y = \infty$

$$\therefore \quad I = \frac{\pi}{8}\int_0^{\infty} \frac{1}{y^2+1}\, dy = \frac{\pi}{8}\left[\tan^{-1} y\right]_0^{\infty} = \frac{\pi}{8}\left[\frac{\pi}{2} - 0\right] = \frac{\pi^2}{16}.$$

OR

Given integral is $\int_0^4 \left(x + e^{2x}\right) dx$.

We know that $\int_a^b f(x)\, dx = \lim_{h\to 0} h\left[f(a) + f(a+h) + f(a+2h) + \ldots + f(a+(n-1)h)\right]$,

where $n = \frac{b-a}{h}$.

Choose $a = 0$, $b = 4$, $f(x) = x + e^{2x}$, then

$$\int_0^4 \left(x + e^{2x}\right) dx = \lim_{h\to 0} h\left[f(0) + f(0+h) + f(0+2h) + \ldots + f(0+(n-1)h)\right] \qquad ...(1)$$

where $n = \frac{4-0}{h} = \frac{4}{h}$. ...(2)

Now, $f(0+hp) = (0+hp) + e^{2(0+hp)} = hp + e^{2hp}$ ...(3)

Putting $p = 0, 1, 2, \ldots, (n-1)$ successively in (3), we get

$$\left\{\begin{array}{ccc} f(0) & = & 1 \\ f(0+h) & = & h + e^{2h} \\ f(0+2h) & = & 2h + e^{4h} \\ \vdots & & \vdots \\ f(0+(n-1)h) & = & (n-1)h + e^{2(n-1)h} \end{array}\right.$$

Adding above 'n' equations, we get

$$f(0) + f(0+h) + f(0+2h) + \ldots + f(0+(n-1)h)$$

$$= \left[1 + e^{2h} + e^{4h} + \ldots + e^{2(n-1)h}\right] + h[1 + 2 + \ldots + (n-1)]$$

$$= \frac{\left(e^{2h}\right)^n - 1}{e^{2h} - 1} + h\left[\frac{(n-1)n}{2}\right]$$

$$= \frac{e^8 - 1}{e^{2h} - 1} + \frac{h}{2}\left(\frac{4}{h} - 1\right)\left(\frac{4}{h}\right) \qquad \text{[Using (2)]}$$

$$= \frac{e^8 - 1}{e^{2h} - 1} + 2\left(\frac{4}{h} - 1\right)$$

$$= \frac{e^8 - 1}{e^{2h} - 1} + \frac{2}{h}(4 - h).$$

Putting this value in (1), we get

$$\int_0^4 (x+e^{2x})\,dx = \lim_{h\to 0} h\left[\frac{e^8-1}{e^{2h}-1}+\frac{2}{h}(4-h)\right]$$

$$= \lim_{h\to 0}\left[(e^8-1)\left(\frac{h}{e^{2h}-1}\right)+2(4-h)\right]$$

$$= \lim_{h\to 0}\left[\left(\frac{e^8-1}{2}\right)\left(\frac{2h}{e^{2h}-1}\right)+2(4-h)\right] = \frac{e^8-1}{2}+8.$$ ■

28. Let a, b, c be the direction ratios of normal to the required plane.

Given that the required plane is perpendicular to the line of intersection of the planes

$$x-y+2z-3=0 \text{ and } 2x-y-3z=0.$$

$\therefore$ Direction ratios of the line of intersection of given planes are a, b, c.

Also, direction ratios of normal to the first plane are $1, -1, 2$

and direction ratios of normal to the second plane are $2, -1, -3$.

$\therefore$ $$1\cdot a - 1\cdot b + 2\cdot c = 0 \quad \text{...(1)}$$

and $$2\cdot a - 1\cdot b - 3\cdot c = 0 \quad \text{...(2)}$$

On solving (1) and (2) by cross multiplication, we get

$$\frac{a}{3+2} = \frac{-b}{-3-4} = \frac{c}{-1+2}$$

i.e., $$\frac{a}{5} = \frac{b}{7} = \frac{c}{1} = k \text{ (say)}$$

$\Rightarrow$ $$a=5k,\ b=7k,\ c=k$$

Also, the required plane passes through the point $(4, -3, 2)$.

Let $x_1 = 4$, $y_1 = -3$, $z_1 = 2$.

Hence, equation of the required plane is

$$a(x-x_1)+b(y-y_1)+c(z-z_1)=0$$

i.e., $$5k(x-4)+7k(y+3)+k(z-2)=0$$

i.e., $$5(x-4)+7(y+3)+(z-2)=0$$

i.e., $$5x+7y+z=1. \quad \text{...(3)}$$

Now, given equation of line is $\vec{r} = \hat{i}+2\hat{j}-\hat{k}+\lambda(\hat{i}+3\hat{j}-9\hat{k})$.

So, equation of the given line in Cartesian form is

$$\frac{x-1}{1} = \frac{y-2}{3} = \frac{z+1}{-9}.$$

Let $\dfrac{x-1}{1} = \dfrac{y-2}{3} = \dfrac{z+1}{-9} = k$.

Then, $x = k+1$, $y = 3k+2$, $z = -9k-1$. ...(4)

From (3) and (4), we get

$$5(k+1)+7(3k+2)+(-9k-1)=1$$

$\Rightarrow$ $$k=-1$$

Putting $k=-1$ in (4), we get $\quad x=0,\ y=-1,\ z=8$.

Hence, the point of intersection of the given line and required plane is $(0, -1, 8)$. ■

29. Let the quantity of Food 1 $= x$ kg

and the quantity of Food 2 $= y$ kg

Let total cost $= ₹\ Z$

We can represent the given L.P.P. in the following tabular form:

	Food 1	**Food 2**	**Requirement**
Cost (₹)	$50x$	$70y$	Minimise
Vitamin A (units)	$2x$	y	At least 8
Vitamin C (units)	x	$2y$	At least 10

Hence, given L.P.P. is, Minimise $Z = 50x + 70y$

subject to the constraints:

$$2x + y \geq 8, \quad x + 2y \geq 10, \quad x \geq 0, \quad y \geq 0$$

We consider the following equations:

$2x + y = 8$

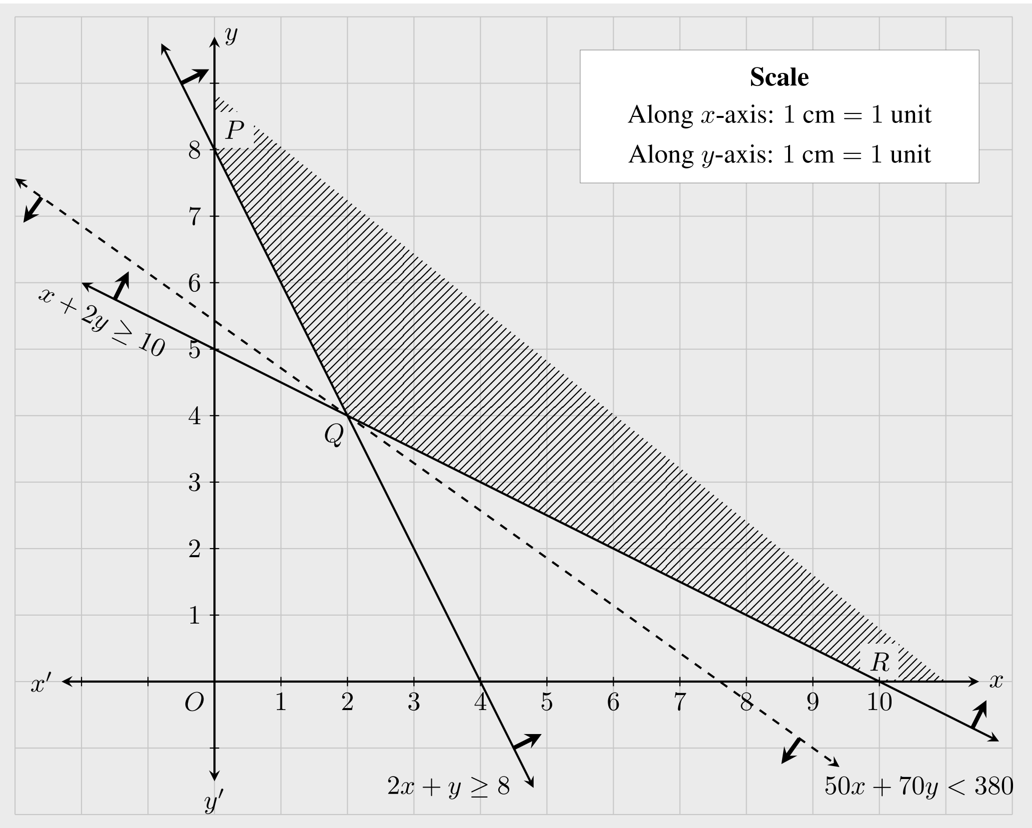
x	0	4
y	8	0

$x + 2y = 10$

x	0	10
y	5	0

$x = 0, \; y = 0$

The feasible region of L.P.P. is unbounded, as shown shaded in the graph.



Corner Points	**Value of $Z (Z = 50x + 70y)$**
$P(0, 8)$	560
$Q(2, 4)$	380
$R(10, 0)$	500

Since, the feasible region is unbounded and 380 is the minimum value of Z at corner points.

So, we consider the open half plane $50x + 70y < 380$, which has no point in common with the feasible region.

$\therefore$ 380 is the minimum value of Z in the feasible region at $x = 2$, $y = 4$.

Hence, quantity of Food $1 = 2$ kg, quantity of Food $2 = 4$ kg and minimum cost = ₹ 380. ■

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Practice Paper – 1

Time: 3 Hours **Max. Marks: 100**

General Instructions

(i) **All** questions are compulsory.

(ii) Please check that this question paper consists of **29** questions.

(iii) Questions **1** to **4** in Section A are Very Short Answer Type Questions carrying **1** mark each.

(iv) Questions **5** to **12** in Section B are Short Answer I Type Questions carrying **2** marks each.

(v) Questions **13** to **23** in Section C are Long Answer I Type Questions carrying **4** marks each.

(vi) Questions **24** to **29** in Section D are Long Answer II Type Questions carrying **6** marks each.

(vii) Please write down the serial number of the question before attempting it.

Section A

Question numbers **1** *to* **4** *carry* **1** *mark each.*

1. Evaluate: $\tan^{-1}\sqrt{3} - \sec^{-1}(-2)$.

2. If A and B are square matrices of order 3 and $|A| = 5$, $|B| = 3$, then find the value of $|3AB|$.

3. Find λ, if the vectors $\lambda\hat{i} + \hat{j} + 2\hat{k}$, $\hat{i} + \lambda\hat{j} - \hat{k}$ and $2\hat{j} - \hat{j} + \lambda\hat{k}$ are coplanar.

4. Let $*$ be a binary operation, on the set of all non-zero real numbers, given by $a * b = \dfrac{ab}{5}$ for all $a, b \in \mathbb{R} - \{0\}$. Find the value of x, given that $2 * (x * 5) = 10$.

Section B

Question numbers **5** *to* **12** *carry* **2** *marks each.*

5. If the curve $ay + x^2 = 7$ and $x^3 = y$ cut orthogonally at $(1, 1)$, then find the value of a.

6. A bag contains 5 red marbles and 3 black marbles. Three marbles are drawn one by one without replacement. What is the probability that at least one of the three marbles drawn be black, if the first marble is red?

7. Prove that: $\cot^{-1}7 + \cot^{-1}8 + \cot^{-1}18 = \cot^{-1}3$.

8. Evaluate: $\displaystyle\int_0^{\pi/2} \frac{\tan x}{1 + m^2 \tan^2 x}\, dx$

9. Prove that $[\,\vec{a} + \vec{b} \;\; \vec{b} + \vec{c} \;\; \vec{c} + \vec{a}\,] = 2[\,\vec{a} \;\; \vec{b} \;\; \vec{c}\,]$.

10. Show that $\begin{vmatrix} \sqrt{23} + \sqrt{3} & \sqrt{5} & \sqrt{5} \\ \sqrt{15} + \sqrt{46} & 5 & \sqrt{10} \\ 3 + \sqrt{115} & \sqrt{15} & 5 \end{vmatrix} = 0.$

11. Find the intervals on which the function $f(x) = \tan^{-1}(\sin x + \cos x)$ on $\left(0, \dfrac{\pi}{4}\right)$ is

(I) strictly increasing or strictly decreasing

(II) increasing or decreasing.

12. Evaluate: $\displaystyle\int \frac{x + \sin x}{1 + \cos x}\, dx$

Section C

Question numbers **13** *to* **23** *carry* **4** *marks each.*

13. Find the area of the quadrilateral $ABCD$, where $A(0,4,1)$, $B(2,3,-1)$, $C(4,5,0)$ and $D(2,6,2)$.

OR

Using vectors, find the value of k such that the points $(k,-10,3)$, $(1,-1,3)$ and $(3,5,3)$ are collinear.

14. Evaluate: $\int \sin^{-1}\sqrt{\dfrac{x}{a+x}}\,dx$

15. If $x^m y^n = (x+y)^{m+n}$, then prove that $\dfrac{dy}{dx} = \dfrac{y}{x}$.

16. Prove, using properties of determinants: $\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & bc & c^2+1 \end{vmatrix} = 1 + a^2 + b^2 + c^2$.

17. Solve: $(x+y)(dx-dy) = dx+dy$.

18. A letter is known to have come either from TATANAGAR or CALCUTTA. On the envelope just two consecutive letters TA are visible. What is the probability that the letter has come from TATANAGAR?

OR

Suppose that 6% of the people with blood group O are left-handed and 10% of those with other blood groups are left-handed. 30% of the people have blood group O. If a left-handed person is selected at random, what is the probability that he/she will have blood group O?

19. Find the angle between the following pair of lines:

$$\frac{-x+2}{-2} = \frac{y-1}{7} = \frac{z+3}{-3} \text{ and } \frac{x+2}{-1} = \frac{2y-8}{4} = \frac{z-5}{4}$$

and check whether the lines are parallel or perpendicular.

20. A spherical ball of salt is dissolving in water in such a manner that the rate of decrease of the volume at any instant is proportional to the surface. Prove that the radius is decreasing at a constant rate.

OR

A ladder, 5 m long, standing on a horizontal floor, leans against a vertical wall. If the top of the ladder slides downwards at the rate of 10 cm/sec, then find the rate at which the angle between the floor and the ladder is decreasing when lower end of ladder is 2 m from the wall.

21. In a hurdle race, a player has to cross 10 hurdles. The probability that he will clear each hurdle is $\dfrac{5}{6}$. What is the probability that he will knock down fewer than 2 hurdles? **What life skills should the player develop to improve his performance?**

22. Evaluate: $\displaystyle\int_0^{\pi} \frac{x}{a^2\cos^2 x + b^2\sin^2 x}\,dx$

23. Find the value of k so that the function $f(x) = \begin{cases} \dfrac{2^{x+2}-16}{4^x-16} & \text{if } x \neq 2 \\ k & \text{if } x = 2 \end{cases}$ is continuous at $x = 2$.

Section D

Question numbers **24** *to* **29** *carry* **6** *marks each.*

24. If $A = \begin{bmatrix} 1 & 2 & 0 \\ -2 & -1 & -2 \\ 0 & -1 & 1 \end{bmatrix}$, find A^{-1}. Using A^{-1}, solve the system of linear equations:

$$x - 2y = 10, \;\; 2x - y - z = 8, \;\; -2y + z = 7.$$

25. Find the area of the region bounded by the parabola $y^2 = 2px$ and $x^2 = 2py$.

26. For real numbers x and y, define xRy if and only if $x - y + \sqrt{2}$ is an irrational number. Determine whether the relation R is reflexive, symmetric and transitive.

OR

Solve: $\sin^{-1}(6x) + \sin^{-1}\left(6\sqrt{3}\,x\right) = -\dfrac{\pi}{2}$.

27. If product of distances of the point $(1, 1, 1)$ from origin and plane $\vec{r} \cdot (\hat{i} - \hat{j} + \hat{k}) = -p$ be 8, then find the value of p.

OR

Find the length and foot of perpendicular from the point $P(7, 14, 5)$ to the plane $2x + 4y - z = 2$. Also, find the image of point P in the plane.

28. An Apache helicopter of enemy is flying along the curve given by $y = x^2 + 7$. A soldier, placed at $(3, 7)$, wants to shoot down the helicopter when it is nearest to him. Find the nearest distance.

OR

Find the area of the greatest rectangle that can be inscribed in an ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

29. A man rides his motorcycle at the speed of 50 km/hour. He has to spend ₹ 2 per km on petrol. If he rides it at a faster speed of 80 km/hour, the petrol cost increases to ₹ 3 per km. He has at most ₹ 120 to spend on petrol and one hour's time. Determine the maximum distance that the man can travel. Express it as a L.P.P. and then solve it.

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Solution to Practice Paper – 1

1. $\tan^{-1}\sqrt{3} - \sec^{-1}(-2) = \tan^{-1}\sqrt{3} - \left[\pi - \sec^{-1}(2)\right]$

$$= \tan^{-1}\left(\tan\frac{\pi}{3}\right) - \pi + \sec^{-1}\left(\sec\frac{\pi}{3}\right)$$

$$= \frac{\pi}{3} - \pi + \frac{\pi}{3} \qquad \left\langle \because \frac{\pi}{3} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right); \frac{\pi}{3} \in [0, \pi] - \left\{\frac{\pi}{2}\right\} \right\rangle$$

$$= -\frac{\pi}{3}.$$

■

2. Given that A and B are square matrices of order 3 such that $|A| = 5$ and $|B| = 3$.

Then, $|3AB| = 3^3|AB|$ $\quad [\because |kA| = k^n A,$ where A is $n \times n$ matrix$]$

$$= 3^3|A||B|$$

$$= (27)(5)(3)$$

$$= 405.$$

■

3. Let $\vec{a} = \lambda\hat{i} + \hat{j} + 2\hat{k}, \vec{b} = \hat{i} + \lambda\hat{j} - \hat{k}$ and $\vec{c} = 2\hat{j} - \hat{j} + \lambda\hat{k}$.

Since, the given vectors are coplanar.

Then, $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$

$$\Rightarrow \begin{vmatrix} \lambda & 1 & 2 \\ 1 & \lambda & -1 \\ 2 & -1 & \lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda(\lambda^2 - 1) - 1(\lambda + 2) + 2(-1 - 2\lambda) = 0$$

$$\Rightarrow \lambda^3 - \lambda - \lambda - 2 - 2 - 4\lambda = 0$$

$$\Rightarrow \lambda^3 - 6\lambda - 4 = 0$$

$$\Rightarrow (\lambda + 2)(\lambda^2 - 2\lambda - 2) = 0$$

$$\Rightarrow \lambda = -2, 1 \pm \sqrt{3}.$$

■

4. Given $*$ on $\mathbb{R} - \{0\}$, defined by $a * b = \dfrac{ab}{5}$.

Given $2 * (x * 5) = 10$

$$\Rightarrow 2 * \left(\frac{5x}{5}\right) = 10$$

$$\Rightarrow 2 * x = 10$$

$$\Rightarrow \frac{2x}{5} = 10$$

$$\Rightarrow x = 25$$

■

5. Given curves are $ay + x^2 = 7$ ...(1)

and $x^3 = y$ ...(2)

Also, given that the point of intersection of the given curves is $(1, 1)$.

Differentiating both sides of (1) and (2) w.r.t. x, we get

$$a\,\frac{dy}{dx} + 2x = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{2x}{a}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(1,1)} = -\frac{2}{a} = m_1 \text{ (say)}$$

$$3x^2 = \frac{dy}{dx}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(1,1)} = 3 = m_2 \text{ (say)}$$

Since, given two curves are orthogonal.

$$\therefore \quad m_1 m_2 = -1$$

$$\Rightarrow \quad \left(-\frac{2}{a}\right)(3) = -1$$

$$\Rightarrow \quad a = 6.$$

■

6. Number of red marbles $= 5$.

Number of black marbles $= 3$.

Total number of marbles $= 8$.

Number of marbles drawn $= 3$ (without replacement).

Let E and F be the events defined as follows:

E : At least one of the marbles is black and F : First marble is red.

Then, $E \cap F$: At least one of the marbles is black and first marble is red.

Also, $P(F) = \dfrac{5}{8}$,

$$P(E \cap F) = P(RRB) + P(RBR) + P(RBB)$$

$$= \left(\frac{5}{8} \times \frac{4}{7} \times \frac{3}{6}\right) + \left(\frac{5}{8} \times \frac{3}{7} \times \frac{4}{6}\right) + \left(\frac{5}{8} \times \frac{3}{7} \times \frac{2}{6}\right) = \frac{25}{56}.$$

Hence, required probability $= P(E|F) = \dfrac{P(E \cap F)}{P(F)} = \dfrac{\left(\frac{25}{56}\right)}{\left(\frac{5}{8}\right)} = \dfrac{5}{7}$. ■

7. L.H.S. $= \cot^{-1}7 + \cot^{-1}8 + \cot^{-1}18$

$$= \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{8} + \tan^{-1}\frac{1}{18}$$

$$= \tan^{-1}\left[\frac{\frac{1}{7} + \frac{1}{8}}{1 - \frac{1}{7} \times \frac{1}{8}}\right] + \tan^{-1}\frac{1}{18} \qquad \left[\because \left(\frac{1}{7}\right)\left(\frac{1}{8}\right) < 1\right]$$

$$= \tan^{-1}\frac{3}{11} + \tan^{-1}\frac{1}{18}$$

$$= \tan^{-1}\left[\frac{\frac{3}{11} + \frac{1}{18}}{1 - \frac{3}{11} \times \frac{1}{18}}\right] \qquad \left[\because \left(\frac{3}{11}\right)\left(\frac{1}{18}\right) < 1\right]$$

$$= \tan^{-1}\frac{1}{3} = \cot^{-1}3 = \text{ R.H.S.}$$

■

8. Let $I = \int_0^{\pi/2} \frac{\tan x}{1+m^2\tan^2 x}\,dx = \int_0^{\pi/2} \frac{\sin x\ \cos x}{\cos^2 x + m^2 \sin^2 x}\,dx.$

Put $\cos^2 x + m^2 \sin^2 x = y$

$\Rightarrow \quad (-2\cos x\ \sin x + 2m^2 \sin x\ \cos x)\,dx = dy$

$\Rightarrow \quad 2\sin x\ \cos x(m^2-1)\,dx = dy$

$\Rightarrow \quad \sin x\ \cos x\,dx = \frac{1}{2(m^2-1)}\,dy$

If $x = 0$, then $y = 1$

If $x = \frac{\pi}{2}$, then $y = m^2$

$$\therefore\quad I = \frac{1}{2(m^2-1)}\int_1^{m^2}\frac{1}{y}\,dy = \frac{1}{2(m^2-1)}\left[\log|y|\right]_1^{m^2} = \frac{1}{2(m^2-1)}\left[\log m^2 - \log 1\right]$$

$$= \frac{1}{2(m^2-1)}[2\log m - 0]$$

$$= \frac{\log m}{m^2-1}.$$

■

9. L.H.S. $= [\,\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}\,]$

$= (\vec{a}+\vec{b})\cdot\{(\vec{b}+\vec{c})\times(\vec{c}+\vec{a})\}$

$= (\vec{a}+\vec{b})\cdot\{(\vec{b}\times\vec{c})+(\vec{b}\times\vec{a})+(\vec{c}\times\vec{c})+(\vec{c}\times\vec{a})\}$

$= (\vec{a}+\vec{b})\cdot\{(\vec{b}\times\vec{c})+(\vec{b}\times\vec{a})+(\vec{c}\times\vec{a})\}$ $\quad [\because \vec{c}\times\vec{c}=\vec{0}\,]$

$= \vec{a}\cdot(\vec{b}\times\vec{c})+\vec{a}\cdot(\vec{b}\times\vec{a})+\vec{a}\cdot(\vec{c}\times\vec{a})+\vec{b}\cdot(\vec{b}\times\vec{c})+\vec{b}\cdot(\vec{b}\times\vec{a})+\vec{b}\cdot(\vec{c}\times\vec{a})$

$= \vec{a}\cdot(\vec{b}\times\vec{c})+0+0+0+0+\vec{a}\cdot(\vec{b}\times\vec{c})$

$= 2\vec{a}\cdot(\vec{b}\times\vec{c})$

$= 2[\,\vec{a}\ \ \vec{b}\ \ \vec{c}\,] =$ R.H.S.

■

10. L.H.S. $= \begin{vmatrix}\sqrt{23}+\sqrt{3} & \sqrt{5} & \sqrt{5}\\ \sqrt{15}+\sqrt{46} & 5 & \sqrt{10}\\ 3+\sqrt{115} & \sqrt{15} & 5\end{vmatrix}$

$= \begin{vmatrix}\sqrt{23}+\sqrt{3} & \sqrt{5} & \sqrt{5}\\ \sqrt{46}+\sqrt{15} & 5 & \sqrt{10}\\ \sqrt{115}+3 & \sqrt{15} & 5\end{vmatrix}$

$= \begin{vmatrix}\sqrt{23} & \sqrt{5} & \sqrt{5}\\ \sqrt{46} & 5 & \sqrt{10}\\ \sqrt{115} & \sqrt{15} & 5\end{vmatrix} + \begin{vmatrix}\sqrt{3} & \sqrt{5} & \sqrt{5}\\ \sqrt{15} & 5 & \sqrt{10}\\ 3 & \sqrt{15} & 5\end{vmatrix}$

$= \sqrt{23}\,\sqrt{5}\begin{vmatrix}1 & \sqrt{5} & 1\\ \sqrt{2} & 5 & \sqrt{2}\\ \sqrt{5} & \sqrt{15} & \sqrt{5}\end{vmatrix} + \sqrt{3}\,\sqrt{5}\begin{vmatrix}1 & 1 & \sqrt{5}\\ \sqrt{5} & \sqrt{5} & \sqrt{10}\\ \sqrt{3} & \sqrt{3} & 5\end{vmatrix}$

[On taking common $\sqrt{23}$ from C_1 and $\sqrt{5}$ from C_3 in first determinant; on taking common $\sqrt{3}$ from C_1 and $\sqrt{5}$ from C_2 in second determinant]

$$= \sqrt{23}\,\sqrt{5}\,(0) + \sqrt{3}\,\sqrt{5}\,(0) \qquad \left[\begin{array}{l}\because C_1 \text{ and } C_3 \text{ are identical in first determinant;}\\ C_1 \text{ and } C_2 \text{ are identical in second determinant}\end{array}\right]$$

$$= 0$$

$$= \text{R.H.S.}$$

■

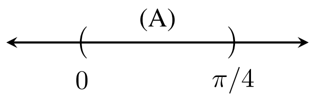
11. Given $f(x) = \tan^{-1}(\sin x + \cos x)$ on $\left(0, \dfrac{\pi}{4}\right)$.

Then, $f'(x) = \dfrac{(\cos x - \sin x)}{1 + (\sin x + \cos x)^2}$.

Now, $f'(x) = 0 \quad \Rightarrow \quad \dfrac{(\cos x - \sin x)}{1 + (\sin x + \cos x)^2} = 0$

$\Rightarrow \quad \cos x - \sin x = 0$

$\Rightarrow \quad \tan x = 1,$ which is not true for any $x \in \left(0, \dfrac{\pi}{4}\right)$.



	Sub-interval	Test Point	Test Value	Sign of $f'(x)$	Conclusion
(A)	$\left(0, \dfrac{\pi}{4}\right)$	$\dfrac{\pi}{6}$	$f'\left(\dfrac{\pi}{6}\right) = \dfrac{2(\sqrt{3}-1)}{4 + (\sqrt{3}+1)^2}$	$(+)$	$f'(x) > 0$

(I) $f(x)$ is strictly increasing on $\left(0, \dfrac{\pi}{4}\right)$.

(II) $f(x)$ is increasing on $\left(0, \dfrac{\pi}{4}\right)$.

■

12. Let $I = \displaystyle\int \frac{x + \sin x}{1 + \cos x}\, dx$

$$= \int \frac{x + 2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\, dx$$

$$= \frac{1}{2}\int \underbrace{x}_{\text{I}}\, \underbrace{\sec^2\frac{x}{2}}_{\text{II}}\, dx + \int \tan\frac{x}{2}\, dx$$

$$= \frac{1}{2}\left\{x\int \sec^2\frac{x}{2}\, dx - \int\left[\frac{d}{dx}(x)\int \sec^2\frac{x}{2}\, dx\right] dx\right\} + \int \tan\frac{x}{2}\, dx$$

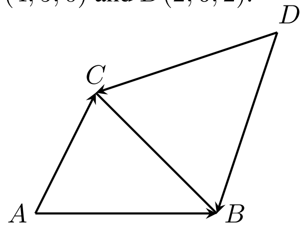
$$= \frac{1}{2}\left\{x\left(2\tan\frac{x}{2}\right) - \int (1)\left(2\tan\frac{x}{2}\right) dx\right\} + \int \tan\frac{x}{2}\, dx$$

$$= x\tan\frac{x}{2} - \int \tan\frac{x}{2}\, dx + \int \tan\frac{x}{2}\, dx$$

$$= x\tan\frac{x}{2} + C.$$

■

13. Given a quadrilateral $ABCD$ with vertices $A(0,4,1)$, $B(2,3,-1)$, $C(4,5,0)$ and $D(2,6,2)$.



Then, $\overrightarrow{AB}$ = Position vector of B − Position vector of A

$$= (2\hat{i}+3\hat{j}-\hat{k})-(0\hat{i}+4\hat{j}+\hat{k}) = 2\hat{i}-\hat{j}-2\hat{k}$$

and $\overrightarrow{AC}$ = Position vector of C − Position vector of A

$$= (4\hat{i}+5\hat{j}+0\hat{k})-(0\hat{i}+4\hat{j}+\hat{k}) = 4\hat{i}+\hat{j}-\hat{k}.$$

Now, $$\overrightarrow{AB}\times\overrightarrow{AC} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & -1 & -2\\ 4 & 1 & -1\end{vmatrix} = \hat{i}(1+2)-\hat{j}(-2+8)+\hat{k}(2+4) = 3\hat{i}-6\hat{j}+6\hat{k}$$

and $|\overrightarrow{AB}\times\overrightarrow{AC}| = \sqrt{(3)^2+(-6)^2+(6)^2} = 9.$

Thus, area of $\Delta ABC = \frac{1}{2}\,|\overrightarrow{AB}\times\overrightarrow{AC}| = \frac{9}{2}$ sq. units.

Also, $\overrightarrow{DB}$ = Position vector of B − Position vector of D

$$= (2\hat{i}+3\hat{j}-\hat{k})-(2\hat{i}+6\hat{j}+2\hat{k}) = 0\hat{i}-3\hat{j}-3\hat{k}$$

and $\overrightarrow{DC}$ = Position vector of C − Position vector of D

$$= (4\hat{i}+5\hat{j}+0\hat{k})-(2\hat{i}+6\hat{j}+2\hat{k}) = 2\hat{i}-\hat{j}-2\hat{k}.$$

Now, $$\overrightarrow{DB}\times\overrightarrow{DC} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 0 & -3 & -3\\ 2 & -1 & -2\end{vmatrix} = \hat{i}(6-3)-\hat{j}(0+6)+\hat{k}(0+6) = 3\hat{i}-6\hat{j}+6\hat{k}$$

and $|\overrightarrow{DB}\times\overrightarrow{DC}| = \sqrt{(3)^2+(-6)^2+(6)^2} = 9.$

Thus, area of $\Delta DBC = \frac{1}{2}\,|\overrightarrow{DB}\times\overrightarrow{DC}| = \frac{9}{2}$ sq. units.

Hence, area of quadrilateral $ABCD$ = area of $\Delta ABC +$ area of $\Delta DBC = 9$ sq. units.

OR

Let the given points be $A(k,-10,3)$, $B(1,-1,3)$ and $C(3,5,3)$.

Then, $\overrightarrow{AB}$ = Position vector of B − Position vector of A

$$= (\hat{i}-\hat{j}+3\hat{k})-(k\hat{i}-10\hat{j}+3\hat{k}) = (1-k)\hat{i}+9\hat{j}+0\hat{k}$$

and $\overrightarrow{BC}$ = Position vector of C − Position vector of B

$$= (3\hat{i}+5\hat{j}+3\hat{k})-(\hat{i}-\hat{j}+3\hat{k}) = 2\hat{i}+6\hat{j}+0\hat{k}.$$

Given that the points A, B and C are collinear.

$\therefore$ Vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are parallel.

$$\Rightarrow \quad \overrightarrow{AB}\times\overrightarrow{BC} = \vec{0}$$

$$\Rightarrow \quad \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1-k & 9 & 0\\ 2 & 6 & 0\end{vmatrix} = \vec{0}$$

$$\Rightarrow \quad \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(6-6k-18) = 0\hat{i} + 0\hat{j} + 0\hat{k}$$

$$\Rightarrow \quad 0\hat{i} + 0\hat{j} + (-6k-12)\hat{k} = 0\hat{i} + 0\hat{j} + 0\hat{k}$$

On comparing both sides, we get

$$-6k - 12 = 0 \quad \Rightarrow \quad k = -2.$$

14. Let $I = \int \sin^{-1}\sqrt{\frac{x}{a+x}}\, dx.$

Put $\sqrt{\frac{x}{a+x}} = \sin y$

$$\Rightarrow \quad \frac{x}{a+x} = \sin^2 y$$

$$\Rightarrow \quad x = (a+x)\ \sin^2 y$$

$$\Rightarrow \quad x(1-\sin^2 y) = a\ \sin^2 y$$

$$\Rightarrow \quad x\ \cos^2 y = a\ \sin^2 y$$

$$\Rightarrow \quad x = a\ \tan^2 y$$

$$\Rightarrow \quad dx = 2a\ \tan y\ \sec^2 y\, dy$$

$$\therefore \quad I = 2a \int \underbrace{y}_{\text{I}}\ \underbrace{\tan y\ \sec^2 y}_{\text{II}}\, dy$$

$$= 2a \left\{ y\left(\frac{\tan^2 y}{2}\right) - \int (1)\left(\frac{\tan^2 y}{2}\right) dy \right\}$$

$$= ay\ \tan^2 y - a\int \tan^2 y\, dy$$

$$= ay\ \tan^2 y - a\int (\sec^2 y - 1)\, dy$$

$$= ay\ \tan^2 y - a\tan y + ay + C$$

$$= a\tan^{-1}\sqrt{\frac{x}{a}}\ \left(\frac{x}{a}\right) - a\ \sqrt{\frac{x}{a}} + a\ \tan^{-1}\sqrt{\frac{x}{a}} + C$$

$$= x\ \tan^{-1}\sqrt{\frac{x}{a}} - \sqrt{ax} + a\ \tan^{-1}\sqrt{\frac{x}{a}} + C.$$

15. Given $x^m y^n = (x+y)^{m+n}$. ...(1)

Differentiating both sides w.r.t. x, we get

$$x^m\ \frac{d}{dx}(y^n) + y^n\ \frac{d}{dx}(x^m) = \frac{d}{dx}(x+y)^{m+n}$$

$$\Rightarrow \quad x^m\ ny^{n-1}\ \frac{dy}{dx} + y^n\ mx^{m-1} = (m+n)\,(x+y)^{m+n-1}\ \frac{d}{dx}(x+y)$$

$$\Rightarrow \quad \frac{n\ x^m\ y^n}{y}\ \frac{dy}{dx} + \frac{m\ x^m\ y^n}{x} = \frac{(m+n)\,(x+y)^{m+n}}{(x+y)}\ \left[1 + \frac{dy}{dx}\right]$$

$$\Rightarrow \quad x^m\ y^n \left[\frac{n}{y}\ \frac{dy}{dx} + \frac{m}{x}\right] = \frac{(m+n)x^m\ y^n}{(x+y)}\ \left[1 + \frac{dy}{dx}\right] \quad \text{[Using (1)]}$$

$$\Rightarrow \quad \frac{n}{y}\frac{dy}{dx} + \frac{m}{x} = \frac{(m+n)}{(x+y)}\left[1 + \frac{dy}{dx}\right]$$

$$\Rightarrow \quad \frac{n}{y}\frac{dy}{dx} + \frac{m}{x} = \frac{(m+n)}{(x+y)} + \frac{(m+n)}{(x+y)}\frac{dy}{dx}$$

$$\Rightarrow \quad \left[\frac{n}{y} - \frac{(m+n)}{(x+y)}\right]\frac{dy}{dx} = \frac{(m+n)}{(x+y)} - \frac{m}{x}$$

$$\Rightarrow \quad \frac{(nx+ny-my-ny)}{y(x+y)}\frac{dy}{dx} = \frac{(mx+nx-mx-my)}{(x+y)x}$$

$$\Rightarrow \quad \frac{(nx-my)}{y}\frac{dy}{dx} = \frac{(nx-my)}{x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x}.$$

Second Method: Given $x^m y^n = (x+y)^{m+n}$.

Taking log on both sides, we get

$$\log(x^m y^n) = \log(x+y)^{m+n}$$

$$\Rightarrow \quad \log x^m + \log y^n = \log(x+y)^{m+n}$$

$$\Rightarrow \quad m\log x + n\log y = (m+n)\log(x+y)$$

Differentiating both sides w.r.t. x, we get

$$m\frac{d}{dx}(\log x) + n\frac{d}{dx}(\log y) = (m+n)\frac{d}{dx}[\log(x+y)]$$

$$\Rightarrow \quad \frac{m}{x} + \frac{n}{y}\frac{dy}{dx} = \frac{(m+n)}{(x+y)}\frac{d}{dx}(x+y)$$

$$\Rightarrow \quad \frac{m}{x} + \frac{n}{y}\frac{dy}{dx} = \frac{(m+n)}{(x+y)}\left[1 + \frac{dy}{dx}\right]$$

Proceeding as in the first method, we get

$$\frac{dy}{dx} = \frac{y}{x}.$$

■

16. L.H.S. $= \begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & bc & c^2+1 \end{vmatrix}$

$$= \frac{abc}{abc}\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & bc & c^2+1 \end{vmatrix}$$

$$= \frac{1}{abc}\begin{vmatrix} a(a^2+1) & a^2b & a^2c \\ ab^2 & b(b^2+1) & b^2c \\ ac^2 & bc^2 & c(c^2+1) \end{vmatrix} \qquad \begin{bmatrix}\text{On multiplying } R_1 \\ \text{by } a, R_2 \text{ by } b \\ \text{and } R_3 \text{ by } c\end{bmatrix}$$

$$= \frac{abc}{abc}\begin{vmatrix} a^2+1 & a^2 & a^2 \\ b^2 & b^2+1 & b^2 \\ c^2 & c^2 & c^2+1 \end{vmatrix} \qquad \begin{bmatrix}\text{On taking common} \\ a \text{ from } C_1, b \text{ from} \\ C_2 \text{ and } c \text{ from } C_3\end{bmatrix}$$

$$= \begin{vmatrix} 1+a^2+b^2+c^2 & 1+a^2+b^2+c^2 & 1+a^2+b^2+c^2 \\ b^2 & b^2+1 & b^2 \\ c^2 & c^2 & c^2+1 \end{vmatrix} \qquad \begin{bmatrix}\text{On applying} \\ R_1 \to R_1 + R_2 + R_3\end{bmatrix}$$

$$= (1+a^2+b^2+c^2)\begin{vmatrix} 1 & 1 & 1 \\ b^2 & b^2+1 & b^2 \\ c^2 & c^2 & c^2+1 \end{vmatrix} \qquad \begin{bmatrix}\text{On taking common} \\ (1+a^2+b^2+c^2) \text{ from } R_1\end{bmatrix}$$

$$= (1+a^2+b^2+c^2)\begin{vmatrix} 1 & 0 & 0 \\ b^2 & 1 & 0 \\ c^2 & 0 & 1 \end{vmatrix} \qquad \begin{bmatrix}\text{On applying} \\ C_2 \to C_2 - C_1 \\ C_3 \to C_3 - C_1\end{bmatrix}$$

$$= (1+a^2+b^2+c^2)\,[1\{1-0\} - 0 + 0] \qquad \begin{bmatrix}\text{On expanding along } R_1\end{bmatrix}$$

$$= 1+a^2+b^2+c^2$$

$$= \text{R.H.S.}$$

Second Method:

$$\text{L.H.S.} = \begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & bc & c^2+1 \end{vmatrix}$$

$$= abc\begin{vmatrix} a+\frac{1}{a} & b & c \\ a & b+\frac{1}{b} & c \\ a & b & c+\frac{1}{c} \end{vmatrix} \qquad \begin{bmatrix}\text{On taking common} \\ a \text{ from } R_1, b \text{ from} \\ R_2 \text{ and } c \text{ from } R_3\end{bmatrix}$$

$$= abc\begin{vmatrix} a+\frac{1}{a} & b & c \\ -\frac{1}{a} & \frac{1}{b} & 0 \\ -\frac{1}{a} & 0 & \frac{1}{c} \end{vmatrix} \qquad \begin{bmatrix}\text{On applying} \\ R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1\end{bmatrix}$$

$$= \begin{vmatrix} a^2+1 & b^2 & c^2 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} \qquad \begin{bmatrix}\text{On multiplying } C_1 \\ \text{by } a, C_2 \text{ by } b \\ \text{and } C_3 \text{ by } c\end{bmatrix}$$

$$= \begin{vmatrix} 1+a^2+b^2+c^2 & b^2 & c^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \qquad \begin{bmatrix}\text{On applying} \\ C_1 \to C_1 + C_2 + C_3\end{bmatrix}$$

$$= (1+a^2+b^2+c^2)\begin{vmatrix} 1 & b^2 & c^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \qquad \begin{bmatrix}\text{On taking common} \\ (1+a^2+b^2+c^2) \text{ from } C_1\end{bmatrix}$$

$$= (1+a^2+b^2+c^2)\,[1\{1-0\} - 0 + 0] \qquad \begin{bmatrix}\text{On expanding along } C_1\end{bmatrix}$$

$$= 1+a^2+b^2+c^2 = \text{R.H.S.}$$

■

17. Given differential equation is

$$(x+y)(dx-dy) = dx+dy$$

$$\Rightarrow \quad (x+y)\,dx-(x+y)\,dy = dx+dy$$

$$\Rightarrow \quad (x+y-1)\,dx = (x+y+1)\,dy$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x+y-1}{x+y+1} \qquad \text{...(1)}$$

Putting $x+y=v$ and $\dfrac{dy}{dx} = \dfrac{dv}{dx} - 1$ in (1), we get

$$\frac{dv}{dx} - 1 = \frac{v-1}{v+1}$$

$$\Rightarrow \quad \frac{dv}{dx} = \frac{v-1}{v+1} + 1$$

$$\Rightarrow \quad \frac{dv}{dx} = \frac{2v}{v+1}$$

$$\Rightarrow \quad \frac{v+1}{v}\,dv = 2\,dx$$

Integrating both sides, we get

$$\int \frac{v+1}{v}\,dv = 2\int 1\,dx$$

$$\Rightarrow \quad \int \left[1+\frac{1}{v}\right] dv = 2\int 1\,dx$$

$$\Rightarrow \quad v + \log|v| = 2x + C$$

$$\Rightarrow \quad x+y+\log|x+y| = 2x+C$$

$$\Rightarrow \quad \log|x+y| = x-y+C,$$

which is the required solution. ■

18. Let B_1, B_2 and E be the events defined as follows:

B_1 : Letter (i.e., message) has come from CALCUTTA

B_2 : Letter (i.e., message) has come from TATANAGAR

and E : Two consecutive letters TA are visible on envelope.

Then, $P(B_1) = \dfrac{1}{2}$, $\quad P(E|B_1) = \dfrac{1}{7}$, $\quad$ [∵ 7 pairs of consecutive letters are CA, AL, LC, CU, UT, TT, **TA**]

$P(B_2) = \dfrac{1}{2}$, $\quad P(E|B_2) = \dfrac{2}{8}$. $\quad$ [∵ 8 pairs of consecutive letters are **TA**, AT, **TA**, AN, NA, AG, GA, AR]

By Bayes' Theorem, we have

$$P(B_2|E) = \frac{P(B_2)P(E|B_2)}{P(B_1)P(E|B_1)+P(B_2)P(E|B_2)} = \frac{\left(\frac{1}{2}\right)\left(\frac{2}{8}\right)}{\left(\frac{1}{2}\right)\left(\frac{1}{7}\right)+\left(\frac{1}{2}\right)\left(\frac{2}{8}\right)} = \frac{7}{11}.$$

Hence, required probability $= \dfrac{7}{11}$.

OR

Let B_1, B_2 and E be the events defined as follows:

B_1 : Person has blood group O, B_2 : Person has blood group other than O

and E : Left-handed person is selected.

Then, $P(B_1) = \frac{30}{100}$, $P(E|B_1) = \frac{6}{100}$,

$P(B_2) = \frac{70}{100}$, $P(E|B_2) = \frac{10}{100}$.

By Bayes' Theorem, we have

$$P(B_1|E) = \frac{P(B_1)P(E|B_1)}{P(B_1)P(E|B_1) + P(B_2)P(E|B_2)}$$

$$= \frac{\left(\frac{30}{100}\right)\left(\frac{6}{100}\right)}{\left(\frac{30}{100}\right)\left(\frac{6}{100}\right) + \left(\frac{70}{100}\right)\left(\frac{10}{100}\right)} = \frac{9}{44}.$$

Hence, required probability $= \frac{9}{44}$. ■

19. Let α be the angle between the given lines.

Equation of first line is

$$\frac{-x+2}{-2} = \frac{y-1}{7} = \frac{z+3}{-3}$$

i.e.,
$$\frac{x-2}{2} = \frac{y-1}{7} = \frac{z+3}{-3}$$

$\therefore$ Direction ratios of first line are 2, 7, -3.

Equation of second line is

$$\frac{x+2}{-1} = \frac{2y-8}{4} = \frac{z-5}{4}$$

i.e.,
$$\frac{x+2}{-1} = \frac{y-4}{2} = \frac{z-5}{4}$$

$\therefore$ Direction ratios of second line are -1, 2, 4.

Let $a_1 = 2$, $b_1 = 7$, $c_1 = -3$ and $a_2 = -1$, $b_2 = 2$, $c_2 = 4$.

Then, $\cos\alpha = \left|\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}\right|$

$$= \left|\frac{(2)(-1) + (7)(2) + (-3)(4)}{\sqrt{(2)^2 + (7)^2 + (-3)^2}\sqrt{(-1)^2 + (2)^2 + (4)^2}}\right| = 0.$$

So, $\alpha = 90°$.

Hence, the given two lines are perpendicular. ■

20. Let R denote the radius, V denote the volume and A denote the surface area of a spherical ball at instant t.

Here, $\frac{dV}{dt} \propto A$; $\frac{dR}{dt} = ?$

Now, $\dfrac{dV}{dt} \propto A$

$\Rightarrow \quad \dfrac{dV}{dt} = -k\,A$ [where k is a positive constant.]

$\Rightarrow \quad \dfrac{dV}{dt} = -4k\pi R^2$ ...(1)

Also, $V = \dfrac{4}{3}\pi R^3$

$\therefore \quad \dfrac{dV}{dt} = 4\pi R^2 \dfrac{dR}{dt}$

$\Rightarrow \quad \dfrac{dR}{dt} = \dfrac{1}{4\pi R^2}\dfrac{dV}{dt}$ ...(2)

From (1) and (2), we get

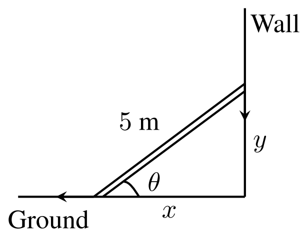
$$\frac{dR}{dt} = \left(\frac{1}{4\pi R^2}\right)(-4k\pi R^2) = -k.$$

Hence, the radius is decreasing at a constant rate.

OR

Let x denote the distance between the foot of ladder and the wall, y denote the distance between the top of ladder and the ground and θ denote the angle between the floor and the ladder at instant t.

Here, $\dfrac{dy}{dt} = -10 \text{ cm/s} = -\dfrac{10}{100} \text{ m/s}; \ \dfrac{d\theta}{dt} = ?$



Now, $\sin\theta = \dfrac{y}{5}$

Differentiating both sides w.r.t. t, we get

$$\cos\theta \frac{d\theta}{dt} = \frac{1}{5}\frac{dy}{dt}$$

$$\Rightarrow \quad \frac{d\theta}{dt} = \frac{1}{5\cos\theta}\frac{dy}{dt}$$

$$\Rightarrow \quad \frac{d\theta}{dt} = \frac{1}{5\left(\frac{x}{5}\right)}\left(-\frac{10}{100}\right)$$

$$\Rightarrow \quad \frac{d\theta}{dt} = -\frac{1}{10x}$$

When $x = 2$, we have $\dfrac{d\theta}{dt} = -\dfrac{1}{20}$ radian/s.

Hence, the required rate of decrease $= \dfrac{1}{20}$ radian/s. ■

21. Suppose that 'Crossing a hurdle' is considered as a trial.

Let success and failure for each trial be defined as follows:

Success : Knocking down a hurdle and Failure : Not knocking down a hurdle.

Since, the trials are independent.

$\therefore$ Probability of failure in each trial, $q = \dfrac{5}{6}$

and probability of success in each trial, $p = 1 - q = \dfrac{1}{6}$.

Number of trials, $n = 10$.

Let X be a random variable defined as the number of successes.

We have, $P(X = x) = {}^nC_x\, p^x\, q^{n-x}$.

Required probability $= P(0$ hurdles are knocked down$) + P(1$ hurdle is knocked down$)$

$$= P(X = 0) + P(X = 1)$$

$$= {}^{10}C_0 \left(\frac{1}{6}\right)^0 \left(\frac{5}{6}\right)^{10} + {}^{10}C_1 \left(\frac{1}{6}\right)^1 \left(\frac{5}{6}\right)^9$$

$$= \left(\frac{5}{6}\right)^{10} + \frac{10}{6}\left(\frac{5}{6}\right)^9 = \frac{5}{2}\left(\frac{5}{6}\right)^9.$$

The life skills the player must develop to improve his performance are hard work, regularity, determination, commitment and sincerity. ■

22. Let $$I = \int_0^{\pi} \frac{x}{a^2\ \cos^2 x + b^2\ \sin^2 x}\, dx \qquad ...(1)$$

$$\Rightarrow \quad I = \int_0^{\pi} \frac{(\pi - x)}{a^2\ \cos^2(\pi - x) + b^2\ \sin^2(\pi - x)}\, dx$$

$$\Rightarrow \quad I = \int_0^{\pi} \frac{(\pi - x)}{a^2\ \cos^2 x + b^2\ \sin^2 x}\, dx \qquad ...(2)$$

On adding (1) and (2), we get

$$2I = \int_0^{\pi} \frac{x}{a^2\ \cos^2 x + b^2\ \sin^2 x}\, dx + \int_0^{\pi} \frac{(\pi - x)}{a^2\ \cos^2 x + b^2\ \sin^2 x}\, dx$$

$$\Rightarrow \quad I = \frac{1}{2}\int_0^{\pi} \frac{(x + \pi - x)}{a^2\ \cos^2 x + b^2\ \sin^2 x}\, dx$$

$$= \frac{\pi}{2}\int_0^{\pi} \frac{1}{a^2\ \cos^2 x + b^2\ \sin^2 x}\, dx$$

Assume $f(x) = \dfrac{1}{a^2\ \cos^2 x + b^2\ \sin^2 x}$.

Then, $f(\pi - x) = \dfrac{1}{a^2\ \cos^2(\pi - x) + b^2\ \sin^2(\pi - x)} = \dfrac{1}{a^2\ \cos^2 x + b^2\ \sin^2 x} = f(x).$

$$\therefore \quad I = \pi \int_0^{\pi/2} \frac{1}{a^2\ \cos^2 x + b^2\ \sin^2 x}\, dx$$

$$= \pi \int_0^{\pi/2} \frac{\sec^2 x}{a^2 + b^2\ \tan^2 x}\, dx \qquad \left[\begin{array}{c}\text{On dividing numerator and}\\ \text{denominator by } \cos^2 x\end{array}\right]$$

Put $\tan x = y$

$\Rightarrow \quad \sec^2 x\, dx = dy$

If $x = 0$, then $y = 0$

If $x = \dfrac{\pi}{2}$, then $y = \infty$

$$\therefore \quad I = \pi \int_0^{\infty} \frac{1}{a^2 + b^2y^2}\, dy = \frac{\pi}{b^2}\int_0^{\infty} \frac{1}{\left(\frac{a}{b}\right)^2 + y^2}\, dy = \frac{\pi}{b^2}\left[\frac{b}{a}\tan^{-1}\left(\frac{by}{a}\right)\right]_0^{\infty}$$

$$= \frac{\pi}{b^2}\left[\frac{\pi b}{2a} - 0\right]$$

$$= \frac{\pi^2}{2ab}.$$ ■

23. Given $f(x) = \begin{cases} \dfrac{2^{x+2} - 16}{4^x - 16} & \text{if } x \neq 2 \\ k & \text{if } x = 2 \end{cases} = \begin{cases} \dfrac{4(2^x - 4)}{(2^x)^2 - 4^2} & \text{if } x \neq 2 \\ k & \text{if } x = 2 \end{cases}$

$$= \begin{cases} \dfrac{4}{2^x + 4} & \text{if } x \neq 2 \\ k & \text{if } x = 2 \end{cases}.$$

Continuity at $x = 2$ We have, $f(2) = k$.

$$\lim_{x \to 2+} [f(x)] = \lim_{x \to 2+} \left[\frac{4}{2^x + 4}\right] = \lim_{h \to 0} \left[\frac{4}{2^{2+h} + 4}\right] \quad \begin{bmatrix}\text{By putting} \\ x = 2 + h\end{bmatrix} = \left[\frac{4}{4+4}\right] = \frac{1}{2}.$$

$$\lim_{x \to 2-} [f(x)] = \lim_{x \to 2-} \left[\frac{4}{2^x + 4}\right] = \lim_{h \to 0} \left[\frac{4}{2^{2-h} + 4}\right] \begin{bmatrix}\text{By putting} \\ x = 2 - h\end{bmatrix} = \left[\frac{4}{4+4}\right] = \frac{1}{2}.$$

So, f is continuous at $x = 2$ if

$$\lim_{x \to 2+} [f(x)] = \lim_{x \to 2-} [f(x)] = f(2)$$

i.e. $$\frac{1}{2} = \frac{1}{2} = k$$

i.e. $$k = \frac{1}{2}$$ ■

24. Given $A = \begin{bmatrix} 1 & 2 & 0 \\ -2 & -1 & -2 \\ 0 & -1 & 1 \end{bmatrix}$.

Then, $|A| = \begin{vmatrix} 1 & 2 & 0 \\ -2 & -1 & -2 \\ 0 & -1 & 1 \end{vmatrix} = 1(-1-2) - 2(-2-0) + 0 = 1 \neq 0.$

$\therefore$ A is non-singular and hence invertible.

Let A_{ij} denote cofactor of a_{ij} in $A = [a_{ij}]$, then

$A_{11} = (-1)^{1+1} \begin{vmatrix} -1 & -2 \\ -1 & 1 \end{vmatrix} = -3,$ $\quad A_{12} = (-1)^{1+2} \begin{vmatrix} -2 & -2 \\ 0 & 1 \end{vmatrix} = 2,$

$A_{13} = (-1)^{1+3} \begin{vmatrix} -2 & -1 \\ 0 & -1 \end{vmatrix} = 2,$ $\quad A_{21} = (-1)^{2+1} \begin{vmatrix} 2 & 0 \\ -1 & 1 \end{vmatrix} = -2,$

$A_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1,$ $\quad A_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix} = 1,$

$A_{31} = (-1)^{3+1} \begin{vmatrix} 2 & 0 \\ -1 & -2 \end{vmatrix} = -4,$ $\quad A_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 0 \\ -2 & -2 \end{vmatrix} = 2,$

$$A_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 2 \\ -2 & -1 \end{vmatrix} = 3.$$

$$\therefore \operatorname{adj} A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} -3 & -2 & -4 \\ 2 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}$$

and $A^{-1} = \dfrac{1}{|A|}(\operatorname{adj} A) = \begin{bmatrix} -3 & -2 & -4 \\ 2 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}$.

Now, given system of equations is

$$x - 2y = 10$$
$$2x - y - z = 8$$
$$-2y + z = 7$$

which can be written in matrix form as

$$\begin{bmatrix} 1 & -2 & 0 \\ 2 & -1 & -1 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 8 \\ 7 \end{bmatrix}$$

i.e., $$A^T X = B,$$

where $A^T = \begin{bmatrix} 1 & -2 & 0 \\ 2 & -1 & -1 \\ 0 & -2 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 10 \\ 8 \\ 7 \end{bmatrix}$.

Now, $|A^T| = |A| = 1 \neq 0$.

So, the given system of equations is consistent and independent, i.e., it has unique solution given by $X = (A^T)^{-1}B = \left(A^{-1}\right)^T B$.

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = X = \left(A^{-1}\right)^T B = \begin{bmatrix} -3 & -2 & -4 \\ 2 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}^T \begin{bmatrix} 10 \\ 8 \\ 7 \end{bmatrix} = \begin{bmatrix} -3 & 2 & 2 \\ -2 & 1 & 1 \\ -4 & 2 & 3 \end{bmatrix} \begin{bmatrix} 10 \\ 8 \\ 7 \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \\ -3 \end{bmatrix}.$$

$\Rightarrow \quad x = 0, y = -5, z = -3.$

Hence, the required solution is $x = 0, y = -5, z = -3$. ■

25. Given curves are

(i) $y^2 = 2px$ $\left(\text{i.e., } y = \pm\sqrt{2p}\sqrt{x}\right)$ (ii) $x^2 = 2py$ $\left(\text{i.e., } y = \dfrac{x^2}{2p}\right)$

To find points of intersection

From (i) and (ii), we have

$$\pm\sqrt{2p}\sqrt{x} = \frac{x^2}{2p}$$

$$\Rightarrow \quad 2px = \frac{x^4}{4p^2}$$

$$\Rightarrow \quad x^4 - 8p^3 x = 0$$

$$\Rightarrow \qquad x = 0, 2p$$

$\therefore$ Curves (i) and (ii) intersect each other at $(0, 0)$ and $(2p, 2p)$.

To plot the required region

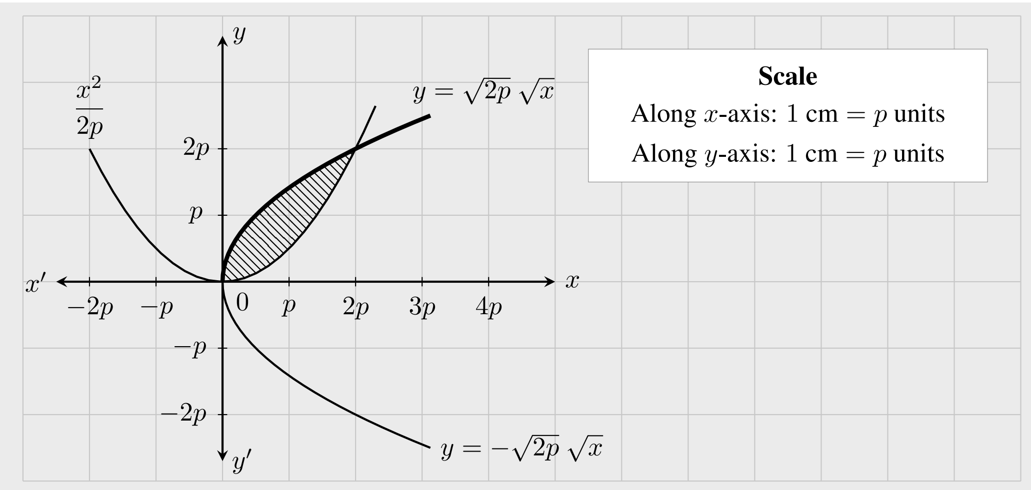
$y = \pm\sqrt{2p}\,\sqrt{x}$

x	0	$2p$	$2p$
y	0	$2p$	$-2p$

$y = \dfrac{x^2}{2p}$

x	0	$2p$	p
y	0	$2p$	$\dfrac{p}{2}$

The required region is shown shaded in the graph.



To find the area of required region

$$A = \int_0^{2p}\left[\sqrt{2p}\,\sqrt{x} - \frac{x^2}{2p}\right]dx = \left[\frac{2\sqrt{2p}\,x^{3/2}}{3} - \frac{x^3}{6p}\right]_0^{2p} = \left(\frac{8p^2}{3} - \frac{8p^2}{6}\right) - (0-0)$$

$$= \frac{4p^2}{3}.$$

Hence, required area $= \dfrac{4p^2}{3}$ sq. units. ■

26. Given $\mathbb{R}$ = Set of all real numbers

and $R = \{(x, y) : x, y \in \mathbb{R}$ and $x - y + \sqrt{2}$ is an irrational number.$\}$.

(i) *Reflexive*: Since, $x - x + \sqrt{2}$ is an irrational number, is true, where $x \in \mathbb{R}$.

$\Rightarrow (x, x) \in R$.

So, R is reflexive.

(ii) *Symmetric*: Let $(x, y) \in R$, where $x, y \in \mathbb{R}$.

$\Rightarrow x - y + \sqrt{2}$ is an irrational number.

$\Rightarrow y - x + \sqrt{2}$ is not an irrational number, for some $x, y \in \mathbb{R}$.

For $x = \sqrt{2}$, $y = 0$, we have $(x, y) \in R$ but $(y, x) \notin R$.

So, R is not symmetric.

(iii) *Transitive*: Let $(x, y) \in R$ and $(y, z) \in R$, where $x, y, z \in \mathbb{R}$.
$\Rightarrow x - y + \sqrt{2}$ is an irrational number and $y - z + \sqrt{2}$ is an irrational number.
$\Rightarrow x - z + \sqrt{2}$ is not an irrational number, for some $x, y, z \in \mathbb{R}$.
For $x = 0$, $y = 2\sqrt{2}$, $z = \sqrt{2}$, we have $(x, y) \in R$, $(y, z) \in R$ but $(x, z) \notin R$.
So, R is not transitive.

OR

Given $\quad \sin^{-1}(6x) + \sin^{-1}\left(6\sqrt{3}\,x\right) = -\dfrac{\pi}{2} \quad$ ...(1)

$$\Rightarrow \quad \sin^{-1}\left(6\sqrt{3}\,x\right) = -\frac{\pi}{2} - \sin^{-1}(6x)$$

$$\Rightarrow \quad 6\sqrt{3}\,x = \sin\left[-\frac{\pi}{2} - \sin^{-1}(6x)\right]$$

$$\Rightarrow \quad 6\sqrt{3}\,x = -\sin\left[\frac{\pi}{2} + \sin^{-1}(6x)\right]$$

$$\Rightarrow \quad 6\sqrt{3}\,x = -\cos\left[\sin^{-1}(6x)\right]$$

Putting $6x = \sin\alpha$, we get

$$\sqrt{3}\,\sin\alpha = -\cos\alpha$$

$$\Rightarrow \quad \sqrt{3}\,\sin\alpha = \pm\sqrt{1 - \sin^2\alpha}$$

$$\Rightarrow \quad \sqrt{3}\,(6x) = \pm\sqrt{1 - (6x)^2}$$

Squaring both sides, we get

$$108x^2 = 1 - 36x^2$$

$$\Rightarrow \quad 144x^2 = 1$$

$$\Rightarrow \quad x = \pm\frac{1}{12}$$

Verification

For $x = \dfrac{1}{12}$, we have [from (1)]

$$\text{L.H.S.} = \sin^{-1}\left(\frac{1}{2}\right) + \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

$$= \frac{\pi}{6} + \frac{\pi}{3}$$

$$= \frac{\pi}{2} \neq \text{R.H.S.}$$

For $x = -\dfrac{1}{12}$, we have [from (1)]

$$\text{L.H.S.} = \sin^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$$

$$= -\sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

$$= -\frac{\pi}{6} - \frac{\pi}{3} = -\frac{\pi}{2} = \text{R.H.S.}$$

Hence, $x = -\dfrac{1}{12}$ is the required solution. ■

27. Given equation of plane is

$$\vec{r} \cdot (\hat{i} - \hat{j} + \hat{k}) = -p$$

i.e.,

$$\vec{r} \cdot (-\hat{i} + \hat{j} - \hat{k}) = p$$

which is of the form

$$\vec{r} \cdot (a\hat{i} + b\hat{j} + c\hat{k}) = d'$$

with $a = -1, b = 1, c = -1, d' = p$.

Also, given point is $(1, 1, 1)$.
Let $x_1 = 1,\ y_1 = 1,\ z_1 = 1$.
Now, distance of the point (x_1, y_1, z_1) from the given plane is given by

$$D_1 = \left|\frac{ax_1 + by_1 + cz_1 - d'}{\sqrt{a^2 + b^2 + c^2}}\right|$$

$$= \left|\frac{(-1)(1) + (1)(1) + (-1)(1) - p}{\sqrt{(-1)^2 + (1)^2 + (-1)^2}}\right| = \left|\frac{-1 - p}{\sqrt{3}}\right| = \left|\frac{1 + p}{\sqrt{3}}\right|.$$

Also, distance between $(1, 1, 1)$ and $(0, 0, 0)$ is

$$D_2 = \sqrt{(0-1)^2 + (0-1)^2 + (0-1)^2} = \sqrt{3}$$

Given that product of these distances is 8.

$\therefore \quad D_1\ D_2 = 8$

$\Rightarrow \quad \left|\frac{1+p}{\sqrt{3}}\right| \sqrt{3} = 8$

$\Rightarrow \quad |1 + p| = 8$

$\Rightarrow \quad 1 + p = \pm\, 8$

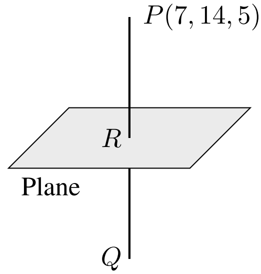
$\Rightarrow \quad p = 7, -9$

Hence, required values of p are 7 and -9.

OR

Given equation of plane is

$$2x + 4y - z = 2 \qquad ...(1)$$



which is of the form

$$ax + by + cz = d'$$

with $a = 2, b = 4, c = -1, d' = 2$.
$\therefore$ Direction ratios of normal to the given plane are $2, 4, -1$.
$\therefore$ Direction ratios of line perpendicular to the given plane are $2, 4, -1$.
So, the equation of line through $(7, 14, 5)$ and perpendicular to the given plane is

$$\frac{x-7}{2} = \frac{y-14}{4} = \frac{z-5}{-1}.$$

Let $\frac{x-7}{2} = \frac{y-14}{4} = \frac{z-5}{-1} = k.$

Then, $x = 2k + 7, y = 4k + 14, z = -k + 5.$ ...(2)

From (1) and (2), we have

$\Rightarrow \quad 2(2k + 7) + 4(4k + 14) - (-k + 5) = 2$

$\Rightarrow \quad 21k + 65 = 2$

$\Rightarrow \quad k = -3$

Putting $k = -3$, in (2), we get $\quad x = 1, y = 2, z = 8$.
$\therefore$ The point of intersection of the line and plane is $(1, 2, 8)$.

$\therefore$ Required foot of perpendicular is $R(1, 2, 8)$.

Now, length of perpendicular = Distance between $(7, 14, 5)$ and $(1, 2, 8)$

$$= \sqrt{(1-7)^2 + (2-14)^2 + (8-5)^2} = \sqrt{189} \text{ units.}$$

Let $Q(\alpha, \beta, \gamma)$ be the image point of the point $P(7, 14, 5)$ in the plane.

Then, $R(1, 2, 8)$ is the mid-point of PQ.

Also, coordinates of mid-point of PQ are $\left(\frac{7+\alpha}{2}, \frac{14+\beta}{2}, \frac{5+\gamma}{2}\right)$.

So, $\frac{7+\alpha}{2} = 1$, $\frac{14+\beta}{2} = 2$, $\frac{5+\gamma}{2} = 8$.

$\therefore \alpha = -5, \beta = -10, \gamma = 11$.

Hence, the image point of P in the given plane is $(-5, -10, 11)$. ■

28. Let D denote the distance between (α, β) on the curve $y = x^2 + 7$ and $(3, 7)$.

$$\Rightarrow \quad D = \sqrt{(\alpha-3)^2 + (\beta-7)^2} \quad \text{...(1)}$$

Also, point (α, β) lies on the curve $y = x^2 + 7$.

$$\Rightarrow \quad \beta = \alpha^2 + 7$$

Putting value of β in (1), we get

$$D = \sqrt{(\alpha-3)^2 + \alpha^4}$$

$$\Rightarrow \quad D = \sqrt{\alpha^4 + \alpha^2 - 6\alpha + 9} \quad \text{...(2)}$$

$$\Rightarrow \quad D^2 = \alpha^4 + \alpha^2 - 6\alpha + 9$$

Putting $D^2 = Z$, we get

$$Z = \alpha^4 + \alpha^2 - 6\alpha + 9$$

Then, $$\frac{dZ}{d\alpha} = 4\alpha^3 + 2\alpha - 6$$

Now, $$\frac{dZ}{d\alpha} = 0 \quad \Rightarrow \quad 4\alpha^3 + 2\alpha - 6 = 0$$

$$\Rightarrow \quad 2\alpha^3 + \alpha - 3 = 0$$

$$\Rightarrow \quad (\alpha - 1)\left(2\alpha^2 + 2\alpha + 3\right) = 0$$

$$\Rightarrow \quad \alpha = 1, \frac{-2 \pm \sqrt{-20}}{4}$$

$$\Rightarrow \quad \alpha = 1$$

Also, $$\frac{d^2Z}{d\alpha^2} = 12\alpha^2 + 2$$

$$\Rightarrow \quad \left(\frac{d^2Z}{d\alpha^2}\right)_{\alpha=1} = 12 + 2 = 14 > 0$$

$\therefore$ By second derivative test, $Z(= D^2)$ is minimum when $\alpha = 1$.

So, D is minimum when $\alpha = 1$.

Putting value of α in (2), we get $D = \sqrt{1 + 1 - 6 + 9} = \sqrt{5}$.

Hence, nearest distance between soldier and helicopter is $\sqrt{5}$ units.

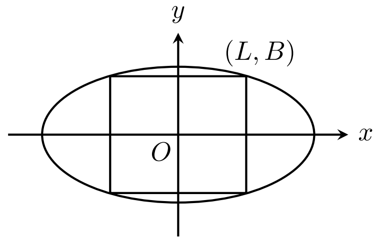
OR

Let A denote the area of rectangle of length $2L$ and breadth $2B$, inscribed in the ellipse having equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Then, $$A = (2L)(2B)$$

$$\Rightarrow \quad A = 4LB \qquad ...(1)$$

Also, point (L, B) lies on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.



$$\therefore \quad \frac{L^2}{a^2} + \frac{B^2}{b^2} = 1$$

$$\Rightarrow \quad \frac{L^2}{a^2} = 1 - \frac{B^2}{b^2}$$

$$\Rightarrow \quad L^2 = \frac{a^2(b^2 - B^2)}{b^2}$$

$$\Rightarrow \quad L = \frac{a}{b}\sqrt{b^2 - B^2} \qquad ...(2)$$

Putting value of L in (1), we get

$$A = 4\left(\frac{a}{b}\sqrt{b^2 - B^2}\right)B$$

$$\Rightarrow \quad A^2 = \frac{16a^2}{b^2}(b^2 - B^2)B^2 = \frac{16a^2}{b^2}\left(b^2B^2 - B^4\right)$$

Putting $A^2 = Z$, we get

$$Z = \frac{16a^2}{b^2}\left(b^2B^2 - B^4\right)$$

Then, $$\frac{dZ}{dB} = \frac{16a^2}{b^2}\left(2b^2B - 4B^3\right)$$

Now, $\frac{dZ}{dB} = 0 \quad \Rightarrow \quad \frac{16a^2}{b^2}\left(2b^2B - 4B^3\right) = 0$

$$\Rightarrow \quad 2b^2B - 4B^3 = 0$$

$$\Rightarrow \quad 2B(b^2 - 2B^2) = 0$$

$$\Rightarrow \quad b^2 - 2B^2 = 0 \qquad (\because B \neq 0)$$

$$\Rightarrow \quad B = \frac{b}{\sqrt{2}} \qquad (\because B > 0)$$

Also, $$\frac{d^2Z}{dB^2} = \frac{16a^2}{b^2}\left(2b^2 - 12B^2\right)$$

$$\Rightarrow \quad \left(\frac{d^2Z}{dB^2}\right)_{B=b/\sqrt{2}} = \frac{16a^2}{b^2}\left[2b^2 - 6b^2\right] = -64a^2 < 0$$

So, by second derivative test, Z is maximum when $B = \frac{b}{\sqrt{2}}$.

Putting value of B in (2), we get

$$L = \frac{a}{b}\sqrt{b^2 - \frac{b^2}{2}} = \frac{a}{\sqrt{2}}.$$

Putting values of L and B in (1), we get $A = 2ab$.

Hence, the area of the required rectangle is $2ab$ sq. units. ■

29. Let the distance travelled at a speed of 50 km/h $= x$ km

and the distance travelled at a speed of 80 km/h $= y$ km

Let total distance travelled $= Z$ km

We can represent the given L.P.P. in the following tabular form:

	Speed 50 km/h	**Speed 80 km/h**	**Requirement**
Distance (km)	x	y	Maximise
Amount spent (₹)	$2x$	$3y$	At most 120
Time spent (hours)	$\frac{x}{50}$	$\frac{y}{80}$	At most 1

Hence, given L.P.P. is, Maximise $Z = x + y$

subject to the constraints:

$$2x + 3y \le 120, \quad \frac{x}{50} + \frac{y}{80} \le 1, \quad x \ge 0, \quad y \ge 0$$

We consider the following equations:

$$2x + 3y = 120$$

x	0	60
y	40	0

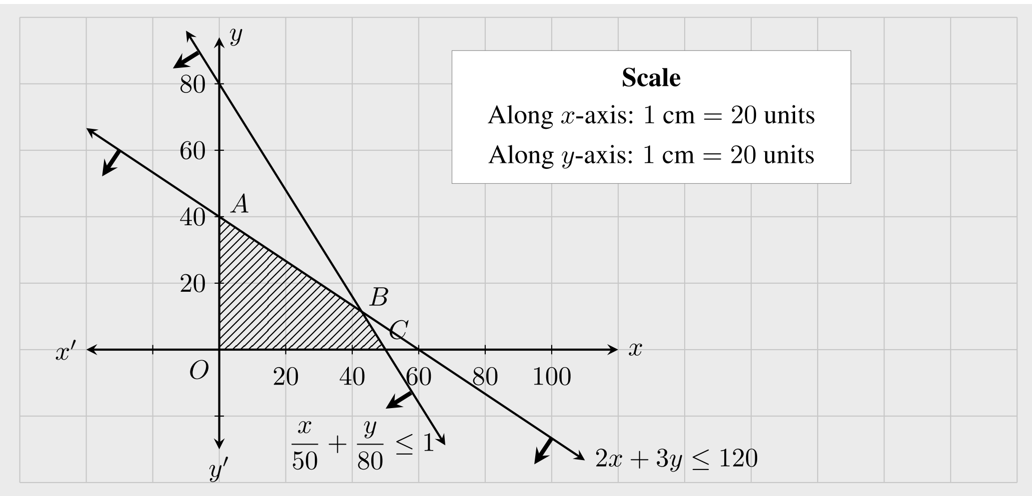
$$\frac{x}{50} + \frac{y}{80} = 1$$

i.e., $8x + 5y = 400$

x	0	50
y	80	0

$$x = 0, y = 0$$

The feasible region of L.P.P. is bounded, as shown shaded in the graph.



Corner Points	**Value of $Z (Z = x + y)$**
$A(0, 40)$	40
$B\left(\frac{300}{7}, \frac{80}{7}\right)$	$\frac{380}{7}$
$C(50, 0)$	50
$O(0, 0)$	0

Since, the feasible region is bounded and $\frac{380}{7}$ is the maximum value of Z at corner points.

$\therefore \frac{380}{7}$ is the maximum value of Z in the feasible region at $x = \frac{300}{7}, y = \frac{80}{7}$.

Hence, distance travelled at a speed of 50 km/h $= \frac{300}{7}$ km,

distance travelled at a speed of 80 km/h $= \frac{80}{7}$ km and maximum distance travelled $= \frac{380}{7}$ km. ■

❑❑❑

Practice Paper – 2

Time: 3 Hours **Max. Marks: 100**

General Instructions

(i) **All** questions are compulsory.

(ii) Please check that this question paper consists of **29** questions.

(iii) Questions **1** to **4** in Section A are Very Short Answer Type Questions carrying **1** mark each.

(iv) Questions **5** to **12** in Section B are Short Answer I Type Questions carrying **2** marks each.

(v) Questions **13** to **23** in Section C are Long Answer I Type Questions carrying **4** marks each.

(vi) Questions **24** to **29** in Section D are Long Answer II Type Questions carrying **6** marks each.

(vii) Please write down the serial number of the question before attempting it.

Section A

Question numbers **1** *to* **4** *carry* **1** *mark each.*

1. If $|\vec{a}| = 4$ and $-3 \leq \lambda \leq 2$, then find the range of $|\lambda\vec{a}|$.

2. Write the function in the simplest form: $\tan^{-1}\left(\dfrac{x}{\sqrt{a^2 - x^2}}\right)$; $|x| < a$.

3. If matrix $A = [a_{ij}]_{2\times 2}$, where $a_{ij} = \begin{cases} 1 & \text{if } i \neq j \\ 0 & \text{if } i = j \end{cases}$, find the value of A^2.

4. Evaluate: $\tan^2(\sec^{-1}2) + \cot^2(\text{cosec}^{-1}3)$.

Section B

Question numbers **5** *to* **12** *carry* **2** *marks each.*

5. Evaluate: $\displaystyle\int \frac{1}{\sin(x-a)\,\sin(x-b)}\,dx$

6. Prove that: $\tan\left[\dfrac{\pi}{4} + \dfrac{1}{2}\cos^{-1}\left(\dfrac{a}{b}\right)\right] + \tan\left[\dfrac{\pi}{4} - \dfrac{1}{2}\cos^{-1}\left(\dfrac{a}{b}\right)\right] = \dfrac{2b}{a}$.

7. For the curve $y = 5x - 2x^3$, if x increases at the rate of 2 units per second, then how fast is the slope of the curve changing when $x = 3$?

8. Solve the differential equation: $\tan y\,\dfrac{dy}{dx} = \cos(x+y) + \cos(x-y)$.

9. Assume that in a family, each child is equally likely to be a boy or a girl. A family with three children is chosen at random. Find the probability that the eldest child is a girl given that the family has at least one girl.

10. Let the function $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \cos x,\ \forall\, x \in \mathbb{R}$. Show that f is neither one-one nor onto.

11. Prove that: $(\vec{a} \times \vec{b})^2 = \vec{a}^{\,2}\,\vec{b}^{\,2} - (\vec{a}\cdot\vec{b})^2$.

12. If $A = \dfrac{1}{\pi}\begin{bmatrix} \sin^{-1}(x\pi) & \tan^{-1}\dfrac{x}{\pi} \\ \sin^{-1}\dfrac{x}{\pi} & \cot^{-1}(\pi x) \end{bmatrix}$, $B = \dfrac{1}{\pi}\begin{bmatrix} -\cos^{-1}(x\pi) & \tan^{-1}\dfrac{x}{\pi} \\ \sin^{-1}\dfrac{x}{\pi} & -\tan^{-1}(\pi x) \end{bmatrix}$, then find the value of $A - B$.

Section C

Question numbers **13** *to* **23** *carry* **4** *marks each.*

13. Evaluate: $\displaystyle\int_0^{\pi} \frac{x\,\sin x}{1+\cos^2 x}\,dx$

14. Prove, using properties of determinants: $\begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = (a-1)^3.$

OR

Express $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$ as the sum of a symmetric and a skew symmetric matrix.

15. The magnitude of the vector product of the vector $\hat{\imath}+\hat{\jmath}+\hat{k}$ with a unit vector along the sum of vectors $2\hat{\imath}+4\hat{\jmath}-5\hat{k}$ and $\lambda\hat{\imath}+2\hat{\jmath}+3\hat{k}$ is equal to $\sqrt{2}$. Find value of λ.

16. For the curve $y = 4x^3 - 2x^5$, find all points at which the tangent passes through the origin.

17. If $x\sqrt{1+y} + y\sqrt{1+x} = 0$, then prove that $\dfrac{dy}{dx} = -\dfrac{1}{(x+1)^2}$.

18. A coin is biased so that the head is 3 times as likely to occur as tail. If the coin is tossed twice, find the probability distribution and expectation of number of tails.

19. Find the equation of the plane passing through the point $(-2, 1, -3)$ and making equal intercepts on the coordinate axes.

OR

A plane meets the coordinate axes in A, B, C such that the centroid of ΔABC is the point (α, β, γ). Show that the equation of the plane is $\dfrac{x}{\alpha} + \dfrac{y}{\beta} + \dfrac{z}{\gamma} = 3$.

20. Prove that the relation R on set $\mathbb{N}\times\mathbb{N}$, defined by

$$(a,b)\ R\ (c,d) \Leftrightarrow a+d = b+c, \text{ for all } (a,b),(c,d) \in \mathbb{N}\times\mathbb{N}$$

is an equivalence relation.

OR

Let $A = \mathbb{R} - \{3\}$ and $B = \mathbb{R} - \{1\}$. Consider the function $f : A \to B$ defined by $f(x) = \dfrac{x-2}{x-3}$. Show that f is one-one and onto and hence find f^{-1}.

21. Evaluate: $\displaystyle\int \frac{\cos 5x + \cos 4x}{1 - 2\cos 3x}\,dx$

22. There are three coins. One is a biased coin that comes up with tail 60% of the times, the second is also a biased coin that comes up heads 75% of the times and third is an unbiased coin. One of the three coins is chosen at random and tossed, it showed heads. What is the probability that it was the unbiased coin? **Is it fair to use a biased coin for toss before the start of a match?**

23. Find the general solution of the differential equation:

$$x\,\frac{dy}{dx} - y = \sqrt{x^2+y^2}.$$

Section D

Question numbers **24** *to* **29** *carry* **6** *marks each.*

24. A window is in the form of a rectangle surmounted by a semicircle. The total perimeter of the window is 10 m. Find the dimensions of the rectangular part of the window to admit maximum light through it.

OR

Show that the radius of the right circular cylinder of greatest curved surface area which can be inscribed in a given cone is equal to half of that of the cone.

25. Find the distance between the point $P(6,5,9)$ and the plane determined by the points $A(3,-1,2)$, $B(5,2,4)$ and $C(-1,-1,6)$.

26. Evaluate: $\displaystyle\int_{-1}^{1} \frac{x^3+|x|+1}{x^2+2|x|+1}\,dx$

27. If $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$, then find A^{-1}. Using A^{-1}, solve the system of equations:

$$2x-3y+5z=11,\;\; 3x+2y-4z=-5,\;\; x+y-2z=-3.$$

OR

If x, y, z are different and $\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$, then show that $1+xyz=0$.

28. A box manufacturer makes large and small boxes from a large piece of cardboard. The large boxes require 4 sq m per box while the small boxes require 3 sq m per box. The manufacturer is required to make at least three large boxes and at most twice as many small boxes as large boxes. If 60 sq m of cardboard is in stock, and if the profits on the large and small boxes are ₹ 3 and ₹ 2 respectively, how many of each should be made in order to maximise the total profit? Formulate the above L.P.P. mathematically and then solve it graphically.

OR

One kind of cake requires 200 g of flour and 25 g of fat, and another kind of cake requires 100 g of flour and 50 g of fat. Find the maximum number of cakes which can be made from 5 kg of flour and 1 kg of fat, assuming that there is no shortage of the other ingredients used in making the cakes. Formulate the above L.P.P. mathematically and then solve it graphically.

29. Find the area bounded by the parabola $y=x^2$ and $y=|x|$.

❑❑❑

Solution to Practice Paper – 2

1. Given $|\vec{a}| = 4$.

Also, $$-3 \le \lambda \le 2 \qquad \text{[Given]}$$

$$\Rightarrow \quad 0 \le |\lambda| \le 3$$

$$\Rightarrow \quad 0 \le |\lambda|\ |\vec{a}| \le 3\ |\vec{a}| \qquad [\because\ |\vec{a}| \ge 0]$$

$$\Rightarrow \quad 0 \le |\lambda \vec{a}| \le 12$$

Hence, the required range is $[0, 12]$. ■

2. Let $y = \tan^{-1}\left(\dfrac{x}{\sqrt{a^2 - x^2}}\right)$.

Putting $x = a\sin\alpha$, we get

$$y = \tan^{-1}\left[\frac{a\sin\alpha}{\sqrt{a^2 - a^2\sin^2\alpha}}\right] = \tan^{-1}\left[\frac{a\sin\alpha}{a\cos\alpha}\right] = \tan^{-1}(\tan\alpha) = \alpha = \sin^{-1}\left(\frac{x}{a}\right).$$

Hence, $\sin^{-1}\left(\dfrac{x}{a}\right)$ is the required simplest form. ■

3. Given $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2\times 2}$, where $a_{ij} = \begin{cases} 1 & \text{if } i \ne j \\ 0 & \text{if } i = j \end{cases}$.

Then, $\quad a_{11} = 0, \quad a_{12} = 1, \quad a_{21} = 1, \quad a_{22} = 0.$

So, $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

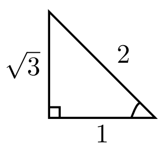
Hence, $A^2 = AA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. ■

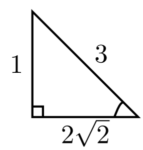
4. $\tan^2\left(\sec^{-1}2\right) + \cot^2\left(\text{cosec}^{-1}3\right)$

$$= \tan^2\left(\tan^{-1}\sqrt{3}\right) + \cot^2\left[\cot^{-1}\left(2\sqrt{2}\right)\right]$$

$$= \left[\tan\left(\tan^{-1}\sqrt{3}\right)\right]^2 + \left\{\cot\left[\cot^{-1}\left(2\sqrt{2}\right)\right]\right\}^2$$

$$= \left[\sqrt{3}\right]^2 + \left[2\sqrt{2}\right]^2 = 11.$$ ■





5. $\displaystyle\int \frac{1}{\sin(x-a)\ \sin(x-b)}\, dx$

$$= \frac{1}{\sin(b-a)} \int \frac{\sin[(x-a)-(x-b)]}{\sin(x-a)\ \sin(x-b)}\, dx$$

$$= \text{cosec}\,(b-a) \int \frac{\sin(x-a)\ \cos(x-b) - \cos(x-a)\ \sin(x-b)}{\sin(x-a)\ \sin(x-b)}\, dx$$

$$= \text{cosec}\,(b-a) \int [\cot(x-b) - \cot(x-a)]\, dx$$

$$= \text{cosec}\,(b-a)\,[\log|\sin(x-b)| - \log|\sin(x-a)|] + C.$$ ■

6. L.H.S. $= \tan\left[\frac{\pi}{4} + \frac{1}{2}\cos^{-1}\left(\frac{a}{b}\right)\right] + \tan\left[\frac{\pi}{4} - \frac{1}{2}\cos^{-1}\left(\frac{a}{b}\right)\right]$.

Putting $\frac{1}{2}\cos^{-1}\left(\frac{a}{b}\right) = \alpha$, we get

L.H.S. $= \tan\left[\frac{\pi}{4} + \alpha\right] + \tan\left[\frac{\pi}{4} - \alpha\right]$

$$= \frac{1 + \tan\alpha}{1 - \tan\alpha} + \frac{1 - \tan\alpha}{1 + \tan\alpha}$$

$$= \frac{(1 + \tan\alpha)^2 + (1 - \tan\alpha)^2}{(1 + \tan\alpha)(1 - \tan\alpha)}$$

$$= \frac{2(1 + \tan^2\alpha)}{1 - \tan^2\alpha}$$

$$= \frac{2}{\cos 2\alpha}$$

$$= \frac{2}{\left(\frac{a}{b}\right)} \qquad \left[\because \frac{1}{2}\cos^{-1}\left(\frac{a}{b}\right) = \alpha \Rightarrow \frac{a}{b} = \cos 2\alpha\right]$$

$$= \frac{2b}{a} = \text{R.H.S.}$$

■

7. Let y denote the ordinate, x denote the abscissa and z denote the slope of the curve at instant t.

Here, $\frac{dx}{dt} = 2$ units/s ; $\frac{dz}{dt} = ?$

Now, $y = 5x - 2x^3$

$\therefore$ $z = \frac{dy}{dx} = 5 - 6x^2$

$\Rightarrow$ $\frac{dz}{dt} = -12x\frac{dx}{dt} = -12x(2) = -24x.$

When $x = 3$, we have $\frac{dz}{dt} = -72$ units/s.

Hence, the required rate of change $= -72$ units/s. ■

8. Given differential equation is

$$\tan y\,\frac{dy}{dx} = \cos(x + y) + \cos(x - y)$$

$$\Rightarrow \tan y\,\frac{dy}{dx} = 2\cos\left[\frac{(x + y) + (x - y)}{2}\right]\cos\left[\frac{(x + y) - (x - y)}{2}\right]$$

$$\Rightarrow \tan y\,\frac{dy}{dx} = 2\cos x\,\cos y$$

$$\Rightarrow \sec y\,\tan y\,dy = 2\cos x\,dx$$

Integrating both sides, we get

$$\int \sec y\,\tan y\,dy = 2\int \cos x\,dx$$

$$\Rightarrow \sec y = 2\sin x + C,$$

which is the required solution. ■

9. Total number of children in a family $= 3$.

Sample space, $S = \{bbb, bbg, bgb, bgg, gbb, gbg, ggb, ggg\}$, where b denotes a boy and g denotes a girl.

Let E and F be the events defined as follows:

E : The eldest child is a girl and F : The family has at least one girl.

Then, $E = \{gbb, gbg, ggb, ggg\}$, $F = \{bbg, bgb, bgg, gbb, gbg, ggb, ggg\}$ and $E \cap F = \{gbb, gbg, ggb, ggg\}$.

Also, $P(F) = \frac{7}{8}$, $P(E \cap F) = \frac{4}{8}$.

Hence, required probability $= P(E|F) = \dfrac{P(E \cap F)}{P(F)} = \dfrac{\left(\frac{4}{8}\right)}{\left(\frac{7}{8}\right)} = \dfrac{4}{7}$. ■

10. Given $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \cos x$.

(i) **One-one:** Let $x_1, x_2 \in \mathbb{R}$ be any two elements.

Then, $f(x_1) = f(x_2)$

$\Rightarrow$ $\cos x_1 = \cos x_2$

Since, $f(0) = f(2\pi)$, but $0 \neq 2\pi$.

So, f is not one-one.

(ii) **Onto:** Let $y \in \mathbb{R}$ be any element.

Then, $f(x) = y$

$\Rightarrow$ $\cos x = y$

For $y = 2 \in \mathbb{R}$, there is no $x \in \mathbb{R}$.

So, f is not onto.

Hence, f is neither one-one nor onto. ■

11. Let θ be the angle between $\vec{a}$ and $\vec{b}$.

We have, $(\vec{a} \times \vec{b})^2 = |\vec{a} \times \vec{b}|^2 = \left(|\vec{a}|\,|\vec{b}|\,\sin\theta\right)^2$

$$= |\vec{a}|^2\,|\vec{b}|^2\,\sin^2\theta$$

$$= |\vec{a}|^2\,|\vec{b}|^2\,(1 - \cos^2\theta)$$

$$= |\vec{a}|^2\,|\vec{b}|^2 - |\vec{a}|^2\,|\vec{b}|^2\,\cos^2\theta$$

$$= |\vec{a}|^2\,|\vec{b}|^2 - \left(|\vec{a}|\,|\vec{b}|\,\cos\theta\right)^2$$

$$= \vec{a}^{\,2}\,\vec{b}^{\,2} - (\vec{a}\cdot\vec{b})^2.$$ ■

12. Given $A = \dfrac{1}{\pi}\begin{bmatrix} \sin^{-1}(x\pi) & \tan^{-1}\dfrac{x}{\pi} \\ \sin^{-1}\dfrac{x}{\pi} & \cot^{-1}(\pi x) \end{bmatrix}$ and $B = \dfrac{1}{\pi}\begin{bmatrix} -\cos^{-1}(x\pi) & \tan^{-1}\dfrac{x}{\pi} \\ \sin^{-1}\dfrac{x}{\pi} & -\tan^{-1}(\pi x) \end{bmatrix}$.

Then, $A - B = \dfrac{1}{\pi}\begin{bmatrix} \sin^{-1}(x\pi) & \tan^{-1}\dfrac{x}{\pi} \\ \sin^{-1}\dfrac{x}{\pi} & \cot^{-1}(\pi x) \end{bmatrix} - \dfrac{1}{\pi}\begin{bmatrix} -\cos^{-1}(x\pi) & \tan^{-1}\dfrac{x}{\pi} \\ \sin^{-1}\dfrac{x}{\pi} & -\tan^{-1}(\pi x) \end{bmatrix}$

$$= \frac{1}{\pi}\left(\begin{bmatrix}\sin^{-1}(x\pi) & \tan^{-1}\dfrac{x}{\pi}\\ \sin^{-1}\dfrac{x}{\pi} & \cot^{-1}(\pi x)\end{bmatrix} - \begin{bmatrix}-\cos^{-1}(x\pi) & \tan^{-1}\dfrac{x}{\pi}\\ \sin^{-1}\dfrac{x}{\pi} & -\tan^{-1}(\pi x)\end{bmatrix}\right)$$

$$= \frac{1}{\pi}\begin{bmatrix}\sin^{-1}(x\pi)+\cos^{-1}(x\pi) & 0\\ 0 & \cot^{-1}(\pi x)+\tan^{-1}(\pi x)\end{bmatrix}$$

$$= \frac{1}{\pi}\begin{bmatrix}\dfrac{\pi}{2} & 0\\ 0 & \dfrac{\pi}{2}\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = \frac{1}{2}\,I. \qquad ■$$

13. Let $$I = \int_0^{\pi} \frac{x\ \sin x}{1+\cos^2 x}\,dx \qquad ...(1)$$

$$\Rightarrow \quad I = \int_0^{\pi} \frac{(\pi - x)\ \sin(\pi - x)}{1+\cos^2(\pi - x)}\,dx$$

$$\Rightarrow \quad I = \int_0^{\pi} \frac{(\pi - x)\ \sin x}{1+\cos^2 x}\,dx \qquad ...(2)$$

On adding (1) and (2), we get

$$2I = \int_0^{\pi} \frac{x\ \sin x}{1+\cos^2 x}\,dx + \int_0^{\pi} \frac{(\pi - x)\ \sin x}{1+\cos^2 x}\,dx$$

$$\Rightarrow \quad 2I = \int_0^{\pi} \frac{(x + \pi - x)\ \sin x}{1+\cos^2 x}\,dx$$

$$\Rightarrow \quad I = \frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx$$

Put $\cos x = y$ | If $x = 0$, then $y = 1$

$\Rightarrow \quad \sin x\,dx = -dy$ | If $x = \pi$, then $y = -1$

$$\therefore \quad I = -\frac{\pi}{2}\int_1^{-1} \frac{1}{1+y^2}\,dy = -\frac{\pi}{2}\left[\tan^{-1} y\right]_1^{-1} = -\frac{\pi}{2}\left[-\frac{\pi}{4} - \frac{\pi}{4}\right] = \frac{\pi^2}{4}. \qquad ■$$

14. $$\text{L.H.S.} = \begin{vmatrix} a^2+2a & 2a+1 & 1\\ 2a+1 & a+2 & 1\\ 3 & 3 & 1\end{vmatrix}$$

$$= \begin{vmatrix} a^2+2a & 2a+1 & 1\\ 1-a^2 & 1-a & 0\\ 3-a^2-2a & 2-2a & 0\end{vmatrix} \qquad \begin{bmatrix}\text{On applying}\\ R_2 \to R_2 - R_1\\ R_3 \to R_3 - R_1\end{bmatrix}$$

$$= \begin{vmatrix} a^2+2a & 2a+1 & 1\\ (1-a)(1+a) & 1-a & 0\\ (3+a)(1-a) & 2(1-a) & 0\end{vmatrix}$$

$$= (1-a)^2 \begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 1+a & 1 & 0 \\ 3+a & 2 & 0 \end{vmatrix}$$ $\left[\begin{matrix}\text{On taking common} \\ (1-a) \text{ from } R_2 \text{ and } R_3\end{matrix}\right]$

$$= (a-1)^2[1\{(2+2a)-(3+a)\}-0+0]$$ $\left[\text{On expanding along } C_3\right]$

$$= (a-1)^3 = \text{R.H.S.}$$

OR

Given $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, then $A^T = \begin{bmatrix} 3 & 1 \\ -4 & -1 \end{bmatrix}$.

Let $P = \frac{1}{2}(A + A^T)$

$$= \frac{1}{2}\left(\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ -4 & -1 \end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix} 6 & -3 \\ -3 & -2 \end{bmatrix} = \begin{bmatrix} 3 & -\frac{3}{2} \\ -\frac{3}{2} & -1 \end{bmatrix}.$$

Since, $P^T = \begin{bmatrix} 3 & -\frac{3}{2} \\ -\frac{3}{2} & -1 \end{bmatrix} = P.$

$\therefore$ P is a symmetric matrix.

Let $Q = \frac{1}{2}(A - A^T)$

$$= \frac{1}{2}\left(\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ -4 & -1 \end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix} 0 & -5 \\ 5 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{5}{2} \\ \frac{5}{2} & 0 \end{bmatrix}.$$

Since, $Q^T = \begin{bmatrix} 0 & \frac{5}{2} \\ -\frac{5}{2} & 0 \end{bmatrix} = -Q.$

$\therefore$ Q is a skew symmetric matrix.

Also, $P + Q = \begin{bmatrix} 3 & -\frac{3}{2} \\ -\frac{3}{2} & -1 \end{bmatrix} + \begin{bmatrix} 0 & -\frac{5}{2} \\ \frac{5}{2} & 0 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} = A.$

Hence, $A = P + Q$, where P is a symmetric matrix and Q is a skew symmetric matrix. ■

15. Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = 2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\vec{c} = \lambda\hat{i} + 2\hat{j} + 3\hat{k}$.

Then, $\vec{b} + \vec{c} = (2\hat{i} + 4\hat{j} - 5\hat{k}) + (\lambda\hat{i} + 2\hat{j} + 3\hat{k}) = (2+\lambda)\hat{i} + 6\hat{j} - 2\hat{k}.$

$\therefore$ Unit vector along the sum of vectors $\vec{b} + \vec{c}$ is given by

$$\hat{r} = \frac{\vec{b} + \vec{c}}{|\vec{b} + \vec{c}|} = \frac{(2+\lambda)\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{(2+\lambda)^2 + (6)^2 + (-2)^2}} = \frac{(2+\lambda)\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{\lambda^2 + 4\lambda + 44}}.$$

Now, $\vec{a} \times \hat{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ \frac{2+\lambda}{\sqrt{\lambda^2+4\lambda+44}} & \frac{6}{\sqrt{\lambda^2+4\lambda+44}} & \frac{-2}{\sqrt{\lambda^2+4\lambda+44}} \end{vmatrix}$

$$= \frac{1}{\sqrt{\lambda^2+4\lambda+44}} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 2+\lambda & 6 & -2 \end{vmatrix}$$

$$= \frac{1}{\sqrt{\lambda^2+4\lambda+44}}\left[\hat{i}(-2-6)-\hat{j}\{-2-(2+\lambda)\}+\hat{k}\{6-(2+\lambda)\}\right]$$

$$= \frac{-8\hat{i}+(4+\lambda)\hat{j}+(4-\lambda)\hat{k}}{\sqrt{\lambda^2+4\lambda+44}}.$$

Given that the magnitude of the vector product of $\vec{a}$ with a unit vector along $\vec{b}+\vec{c}$ is equal to $\sqrt{2}$.

$$\therefore \quad |\vec{a}\times\hat{r}| = \sqrt{2}$$

$$\Rightarrow \quad \left|\frac{-8\hat{i}+(4+\lambda)\hat{j}+(4-\lambda)\hat{k}}{\sqrt{\lambda^2+4\lambda+44}}\right| = \sqrt{2}$$

$$\Rightarrow \quad \frac{\left|-8\hat{i}+(4+\lambda)\hat{j}+(4-\lambda)\hat{k}\right|}{\sqrt{\lambda^2+4\lambda+44}} = \sqrt{2}$$

$$\Rightarrow \quad \sqrt{(-8)^2+(4+\lambda)^2+(4-\lambda)^2} = \sqrt{2}\,\sqrt{\lambda^2+4\lambda+44}$$

$$\Rightarrow \quad 64+(16+\lambda^2+8\lambda)+(16+\lambda^2-8\lambda) = 2(\lambda^2+4\lambda+44)$$

$$\Rightarrow \quad 2\lambda^2+96 = 2\lambda^2+8\lambda+88$$

$$\Rightarrow \quad 8 = 8\lambda$$

$$\Rightarrow \quad \lambda = 1.$$

■

16. Given curve is $y = 4x^3 - 2x^5$. ...(1)

Let the required point on the given curve be (α, β).

$$\therefore \quad \beta = 4\alpha^3 - 2\alpha^5 \qquad ...(2)$$

Differentiating both sides of (1) w.r.t. x, we get

$$\frac{dy}{dx} = 12x^2 - 10x^4$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(\alpha,\beta)} = 12\alpha^2 - 10\alpha^4$$

Now, equation of tangent at (α, β) is $\quad y-\beta = \left(\frac{dy}{dx}\right)_{(\alpha,\beta)}(x-\alpha)$

$$\text{i.e.,} \quad y-\beta = \left(12\alpha^2-10\alpha^4\right)(x-\alpha) \qquad ...(3)$$

Given that tangent passes through the point $(0, 0)$.

Putting $x = 0$ and $y = 0$ in (3), we get

$$0-\beta = \left(12\alpha^2-10\alpha^4\right)(0-\alpha)$$

$$\Rightarrow \quad \beta = 12\alpha^3 - 10\alpha^5 \qquad ...(4)$$

From (2) and (4), we get

$$4\alpha^3 - 2\alpha^5 = 12\alpha^3 - 10\alpha^5$$

$$\Rightarrow \quad 8\alpha^5 - 8\alpha^3 = 0$$

$$\Rightarrow \quad 8\alpha^3(\alpha^2-1) = 0$$

$$\Rightarrow \quad \alpha = 0, 1, -1$$

Putting $\alpha = 0$ in (2), we get $\beta = 0$.

Putting $\alpha = 1$ in (2), we get $\beta = 2$.

Putting $\alpha = -1$ in (2), we get $\beta = -2$.

$\therefore$ $(0, 0)$, $(1, 2)$ and $(-1, -2)$ are the points on the curve.

Also, all the three points satisfy (4).

Hence, required points are $(0, 0)$, $(1, 2)$ and $(-1, -2)$. ■

17. Given $x\sqrt{1+y} + y\sqrt{1+x} = 0$

$\Rightarrow \quad x\sqrt{1+y} = -y\sqrt{1+x}$

$\Rightarrow \quad x^2(1+y) = y^2(1+x)$

$\Rightarrow \quad x^2 + x^2y = y^2 + xy^2$

$\Rightarrow \quad x^2 - y^2 = xy^2 - x^2y$

$\Rightarrow \quad (x+y)(x-y) = -xy(x-y)$

$\Rightarrow \quad x + y = -xy$

$\Rightarrow \quad y(1+x) = -x$

$\Rightarrow \quad y = -\dfrac{x}{1+x}$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = -\left[\frac{(1+x)\dfrac{d}{dx}(x) - (x)\dfrac{d}{dx}(1+x)}{(1+x)^2}\right] = -\left[\frac{(1+x)-(x)}{(1+x)^2}\right] = -\frac{1}{(x+1)^2}.$$ ■

18. Number of times a coin is tossed $= 2$.

Suppose that 'getting a tail' is considered as success.

Given that the head is 3 times as likely to occur as tail.

i.e., $P(\text{a head}) = 3\,P(\text{a tail})$

$\Rightarrow \quad 1 - P(\text{a tail}) = 3\,P(\text{a tail})$

$\Rightarrow \quad P(\text{a tail}) = \dfrac{1}{4}$

So, $P(\text{a tail}) = \dfrac{1}{4}$ and $P(\text{a head}) = \dfrac{3}{4}$.

Let X be a random variable defined as the number of successes.

Then, X can attain the values 0, 1, 2.

Now, $P(X = 0) = P(\text{both are heads})$ [Fix Case]

$= \dfrac{3}{4} \times \dfrac{3}{4} = \dfrac{9}{16}$.

$P(X = 1) = P(\text{1 head and 1 tail})$ [Unfix Case]

$= P(\text{first is head and second is tail}) \times \dfrac{2!}{1!\,1!} = \dfrac{3}{4} \times \dfrac{1}{4} \times \dfrac{2!}{1!\,1!} = \dfrac{6}{16}$.

$P(X = 2) = P(\text{both are tails})$ [Fix Case]

$= \dfrac{1}{4} \times \dfrac{1}{4} = \dfrac{1}{16}$.

So, probability distribution of X is given by

X	0	1	2
$P(X)$	$\frac{9}{16}$	$\frac{6}{16}$	$\frac{1}{16}$

Calculation of Expectation

X	$P(X)$	$X\ P(X)$
0	$\frac{9}{16}$	0
1	$\frac{6}{16}$	$\frac{6}{16}$
2	$\frac{1}{16}$	$\frac{2}{16}$
		$\sum X\ P(X) = \frac{8}{16} = \frac{1}{2}$

Hence, expectation $= E(X) = \sum X\ P(X) = \frac{1}{2}$. ■

19. Given, a plane which makes equal intercepts (say, a) on the coordinate axes.

Then, equation of plane, having intercepts a, a and a on the coordinate axes, is

$$\frac{x}{a} + \frac{y}{a} + \frac{z}{a} = 1$$

$$\Rightarrow \quad x + y + z = a \quad \text{...(1)}$$

Also, given that the plane passes through the point $(-2, 1, -3)$.

$$\therefore \quad -2 + 1 - 3 = a$$

$$\Rightarrow \quad a = -4$$

Putting $a = -4$ in (1), we get

$$x + y + z = -4,$$

which is the required equation.

OR

Equation of plane, having intercepts a, b, c on the coordinate axes, is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \quad \text{...(1)}$$

Given that plane meets the coordinate axes in A, B, C.

$\therefore$ The coordinates of A, B, C are $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$ respectively.

Then, the centroid of the ΔABC is $\left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3}\right)$.

Given that centroid of the ΔABC is (α, β, γ).

$\therefore \quad \frac{a}{3} = \alpha$, $\frac{b}{3} = \beta$, $\frac{c}{3} = \gamma$

i.e., $\quad a = 3\alpha$, $b = 3\beta$, $c = 3\gamma$

Putting values of a, b, c in (1), we get the equation of the required plane as

$$\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 3.$$ ■

20. Given $\mathbb{N}$ = Set of all natural numbers

and $R = \{((a, b), (c, d)) : a + d = b + c; (a, b), (c, d) \in \mathbb{N} \times \mathbb{N}\}$

$= \{((a, b), (c, d)) : a - b = c - d; (a, b), (c, d) \in \mathbb{N} \times \mathbb{N}\}$.

(i) *Reflexive*: Since, $a - b = a - b$, is true, where $(a, b) \in \mathbb{N} \times \mathbb{N}$.

$\Rightarrow ((a, b), (a, b)) \in R$.

So, R is reflexive.

(ii) *Symmetric*: Let $((a, b), (c, d)) \in R$, where $(a, b), (c, d) \in \mathbb{N} \times \mathbb{N}$.

$\Rightarrow a - b = c - d$.

$\Rightarrow c - d = a - b$.

$\Rightarrow ((c, d), (a, b)) \in R$.

So, R is symmetric.

(iii) *Transitive*:

Let $((a, b), (c, d)) \in R$ and $((c, d), (e, f)) \in R$, where $(a, b), (c, d), (e, f) \in \mathbb{N} \times \mathbb{N}$.

$\Rightarrow a - b = c - d$ and $c - d = e - f$.

$\Rightarrow a - b = e - f$.

$\Rightarrow ((a, b), (e, f)) \in R$.

So, R is transitive.

Hence, R is an equivalence relation.

OR

Given $f : \mathbb{R} - \{3\} \to \mathbb{R} - \{1\}$ defined by $f(x) = \dfrac{x-2}{x-3}$.

(i) **One-one:** Let $x_1, x_2 \in \mathbb{R} - \{3\}$ be any two elements.

Then, $$f(x_1) = f(x_2)$$

$$\Rightarrow \quad \frac{x_1 - 2}{x_1 - 3} = \frac{x_2 - 2}{x_2 - 3}$$

$$\Rightarrow \quad x_1x_2 - 2x_2 - 3x_1 + 6 = x_1x_2 - 3x_2 - 2x_1 + 6$$

$$\Rightarrow \quad x_1 = x_2$$

So, f is one-one.

(ii) **Onto:** Let $y \in \mathbb{R} - \{1\}$ be any element.

Then, $$f(x) = y$$

$$\Rightarrow \quad \frac{x-2}{x-3} = y$$

$$\Rightarrow \quad x - 2 = xy - 3y$$

$$\Rightarrow \quad x(1 - y) = 2 - 3y$$

$$\Rightarrow \quad x = \frac{2 - 3y}{1 - y} \qquad \text{...(1)}$$

For every $y \in \mathbb{R} - \{1\}$, we have $x = \dfrac{2-3y}{1-y} \in \mathbb{R} - \{3\}$.

So, f is onto.

Thus, f is bijection and hence invertible.

So, $f^{-1} : \mathbb{R} - \{1\} \to \mathbb{R} - \{3\}$ exists and from (1), we have $x = \dfrac{2-3y}{1-y}$.

$\Rightarrow \; f^{-1}(y) = \dfrac{2-3y}{1-y}$. $\quad [\because f(x) = y \Leftrightarrow x = f^{-1}(y)]$

Hence, f^{-1} is given by $f^{-1}(x) = \dfrac{2-3x}{1-x}$. ■

21. $$\int \frac{\cos 5x + \cos 4x}{1 - 2\cos 3x}\, dx = \int \frac{2\cos\frac{9x}{2}\;\cos\frac{x}{2}}{1 - 2\left[2\cos^2\frac{3x}{2} - 1\right]}\, dx$$

$$= \int \frac{2\cos 3\left(\frac{3x}{2}\right)\;\cos\frac{x}{2}}{3 - 4\cos^2\frac{3x}{2}}\, dx$$

$$= \int \frac{2\left[4\cos^3\frac{3x}{2} - 3\cos\frac{3x}{2}\right]\cos\frac{x}{2}}{3 - 4\cos^2\frac{3x}{2}}\, dx$$

$$= \int \frac{-2\cos\frac{3x}{2}\left[3 - 4\cos^2\frac{3x}{2}\right]\cos\frac{x}{2}}{3 - 4\cos^2\frac{3x}{2}}\, dx$$

$$= -\int 2\cos\frac{3x}{2}\;\cos\frac{x}{2}\, dx$$

$$= -\int [\cos 2x + \cos x]\, dx = -\left[\frac{\sin 2x}{2} + \sin x\right] + C.$$ ■

22. Let B_1, B_2, B_3 and E be the events defined as follows:

B_1 : First biased coin is tossed, $\quad B_2$: Second biased coin is tossed,

B_3 : Unbiased coin is tossed $\quad$ and E : Coin shows head.

Then, $P(B_1) = \dfrac{1}{3}$, $\quad P(E|B_1) = \dfrac{40}{100}$,

$P(B_2) = \dfrac{1}{3}$, $\quad P(E|B_2) = \dfrac{75}{100}$,

$P(B_3) = \dfrac{1}{3}$, $\quad P(E|B_3) = \dfrac{1}{2}$.

By Bayes' Theorem, we have

$$P(B_3|E) = \frac{P(B_3)P(E|B_3)}{P(B_1)P(E|B_1) + P(B_2)P(E|B_2) + P(B_3)P(E|B_3)}$$

$$= \frac{\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)}{\left(\frac{1}{3}\right)\left(\frac{40}{100}\right) + \left(\frac{1}{3}\right)\left(\frac{75}{100}\right) + \left(\frac{1}{3}\right)\left(\frac{1}{2}\right)} = \frac{10}{33}.$$

Hence, required probability $= \dfrac{10}{33}$.

No, it is not fair to use a biased coin for toss before the start of a match. ■

23. Given differential equation is

$$x\,\frac{dy}{dx} - y = \sqrt{x^2+y^2}$$

$$\Rightarrow \qquad x\,\frac{dy}{dx} = y + \sqrt{x^2+y^2}$$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{y}{x} + \frac{\sqrt{x^2+y^2}}{x}$$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{y}{x} + \sqrt{1+\left(\frac{y}{x}\right)^2} \qquad ...(1)$$

Putting $y = vx$ and $\dfrac{dy}{dx} = v + x\,\dfrac{dv}{dx}$ in (1), we get

$$v + x\,\frac{dv}{dx} = v + \sqrt{1+v^2}$$

$$\Rightarrow \qquad x\,\frac{dv}{dx} = \sqrt{1+v^2}$$

$$\Rightarrow \qquad \frac{1}{\sqrt{1+v^2}}\,dv = \frac{1}{x}\,dx$$

Integrating both sides, we get

$$\int \frac{1}{\sqrt{1+v^2}}\,dv = \int \frac{1}{x}\,dx$$

$$\Rightarrow \qquad \log\left|v+\sqrt{1+v^2}\right| = \log|x| + C$$

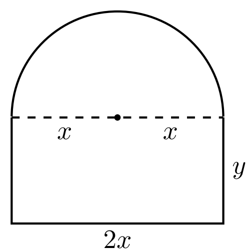
$$\Rightarrow \qquad \log\left|\frac{y}{x}+\sqrt{1+\left(\frac{y}{x}\right)^2}\right| = \log|x| + C,$$

which is the required solution. ■

24. We know that a window with largest area will admit maximum light.

Let A denote the area of the window consisting of a rectangle of length $2x$ and breadth y surmounted by a semicircle of diameter $2x$.

$$\Rightarrow \qquad A = \frac{\pi}{2}\,x^2 + 2xy \qquad ...(1)$$



Now, perimeter of window, $P = 10$

$$\Rightarrow \qquad \pi x + 2x + 2y = 10$$

$$\Rightarrow \qquad y = \frac{10-(\pi+2)x}{2} \qquad ...(2)$$

Putting value of y in (1), we get

$$A = \frac{\pi}{2}\,x^2 + x[10-(\pi+2)x] = \frac{\pi}{2}\,x^2 + 10x - (\pi+2)x^2.$$

Then, $\dfrac{dA}{dx} = \pi x + 10 - 2(\pi+2)x$

Now, $\dfrac{dA}{dx} = 0 \quad \Rightarrow \quad \pi x + 10 - 2(\pi+2)x = 0$

$$\Rightarrow \qquad x = \frac{10}{\pi+4}$$

Also, $$\frac{d^2A}{dx^2} = -(\pi+4)$$

$$\Rightarrow \quad \left(\frac{d^2A}{dx^2}\right)_{x=10/(\pi+4)} = -(\pi+4) < 0$$

$\therefore$ By second derivative test, A is maximum when $x = \dfrac{10}{\pi+4}$.

Putting value of x in (2), we get

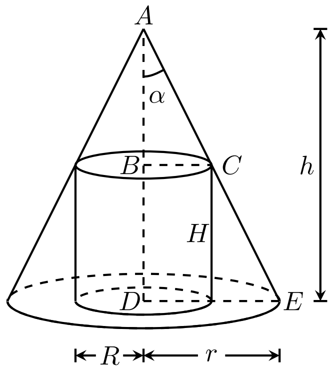
$$y = \frac{10-(\pi+2)\left[\dfrac{10}{\pi+4}\right]}{2} = \frac{10}{\pi+4}.$$

Hence, length and breadth of rectangular part are $\dfrac{20}{\pi+4}$ m and $\dfrac{10}{\pi+4}$ m respectively.

OR

Let A denote the curved surface area of cylinder of radius R and height H, inscribed in a given cone of height h, radius r and semi-vertical angle α.

$$\Rightarrow \quad A = 2\pi RH \qquad ...(1)$$



Also, in right angled ΔABC, we have

$$\frac{BC}{AB} = \tan\alpha$$

$$\Rightarrow \quad \frac{R}{h-H} = \tan\alpha$$

$$\Rightarrow \quad h - H = R\cot\alpha$$

$$\Rightarrow \quad H = h - R\cot\alpha$$

Putting value of H in (1), we get

$$A = 2\pi R(h - R\cot\alpha)$$

$$\Rightarrow \quad A = 2\pi\left(Rh - R^2\cot\alpha\right) \qquad ...(2)$$

Then, $\dfrac{dA}{dR} = 2\pi\,(h - 2R\cot\alpha)$.

Now, $\dfrac{dA}{dR} = 0 \quad \Rightarrow \quad 2\pi\,(h-2R\cot\alpha) = 0$

$$\Rightarrow \quad h - 2R\cot\alpha = 0$$

$$\Rightarrow \quad R = \frac{h\tan\alpha}{2}$$

Also, $$\frac{d^2A}{dR^2} = 2\pi\,(-2\cot\alpha) = -4\pi\cot\alpha$$

$$\Rightarrow \quad \left(\frac{d^2A}{dR^2}\right)_{R=h\tan\alpha/2} = -4\pi\cot\alpha < 0 \qquad \left[\because 0 < \alpha < \frac{\pi}{2}\right]$$

So, by second derivative test, A is maximum when $R = \dfrac{h\tan\alpha}{2}$.

Also, in right angled ΔADE, we have

$$\frac{DE}{AD} = \tan\alpha$$

$$\Rightarrow \qquad \frac{r}{h} = \tan\alpha$$

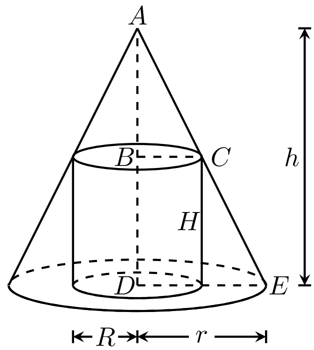
$$\Rightarrow \qquad r = h\tan\alpha$$

Thus, $R = \dfrac{h\tan\alpha}{2} = \dfrac{r}{2}$.

Hence, required radius of the right circular cylinder is equal to half of the radius of the cone.

Second Method:

Let A denote the curved surface area of cylinder of radius R and height H, inscribed in a given cone of height h and radius r.



$$\Rightarrow \qquad A = 2\pi RH \qquad ...(1)$$

Also, $\Delta ABC \sim \Delta ADE$.

$$\Rightarrow \qquad \frac{AB}{AD} = \frac{BC}{DE}$$

$$\Rightarrow \qquad \frac{h-H}{h} = \frac{R}{r}$$

$$\Rightarrow \qquad H = h - \frac{hR}{r}$$

Putting value of H in (1), we get

$$A = 2\pi R\left(h - \frac{hR}{r}\right) = 2\pi hR - \frac{2\pi hR^2}{r}.$$

Then, $\dfrac{dA}{dR} = 2\pi h - \dfrac{4\pi hR}{r}$.

Now, $\dfrac{dA}{dR} = 0 \quad \Rightarrow \quad 2\pi h - \dfrac{4\pi hR}{r} = 0$

$$\Rightarrow \qquad 2\pi h\left(1 - \frac{2R}{r}\right) = 0$$

$$\Rightarrow \qquad 1 - \frac{2R}{r} = 0 \qquad [\because h \neq 0]$$

$$\Rightarrow \qquad R = \frac{r}{2}$$

Also, $\dfrac{d^2A}{dR^2} = -\dfrac{4\pi h}{r}$

$$\Rightarrow \qquad \left(\frac{d^2A}{dR^2}\right)_{R=r/2} = -\frac{4\pi h}{r} < 0 \qquad [\because r, h > 0]$$

So, by second derivative test, A is maximum when $R = \dfrac{r}{2}$.

Hence, required radius of the right circular cylinder is equal to half of the radius of the cone. ■

25. Given points on plane are $A(3,-1,2)$, $B(5,2,4)$ and $C(-1,-1,6)$.

Let $x_1 = 3$, $y_1 = -1$, $z_1 = 2$; $x_2 = 5$, $y_2 = 2$, $z_2 = 4$; $x_3 = -1$, $y_3 = -1$, $z_3 = 6$.

Then, equation of plane through A, B and C is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} x-3 & y+1 & z-2 \\ 2 & 3 & 2 \\ -4 & 0 & 4 \end{vmatrix} = 0$$

$$\Rightarrow \quad (x-3)(12-0)-(y+1)(8+8)+(z-2)(0+12) = 0$$

$$\Rightarrow \quad 12x-16y+12z-76 = 0$$

$$\Rightarrow \quad 3x-4y+3z-19 = 0$$

$$\Rightarrow \quad 3x-4y+3z = 19$$

which is of the form

$$ax+by+cz = d'$$

with $a = 3$, $b = -4$, $c = 3$, $d' = 19$.

Also, given point is $P(6,5,9)$.

Let $x_1 = 6$, $y_1 = 5$, $z_1 = 9$.

Now, distance of the point (x_1, y_1, z_1) from the given plane is

$$D = \left| \frac{ax_1+by_1+cz_1-d'}{\sqrt{a^2+b^2+c^2}} \right|$$

$$= \left| \frac{(3)(6)+(-4)(5)+(3)(9)-19}{\sqrt{(3)^2+(-4)^2+(3)^2}} \right| = \frac{6}{\sqrt{34}}.$$

Hence, required distance $= \dfrac{6}{\sqrt{34}}$ units. ■

26. Let $I = \displaystyle\int_{-1}^{1} \frac{x^3+|x|+1}{x^2+2|x|+1}\,dx$

$$= \int_{-1}^{1} \frac{x^3}{x^2+2|x|+1}\,dx + \int_{-1}^{1} \frac{|x|+1}{x^2+2|x|+1}\,dx$$

$$= I_1 + I_2 \qquad \text{...(1)}$$

Now, $I_1 = \displaystyle\int_{-1}^{1} \frac{x^3}{x^2+2|x|+1}\,dx$

Assume $f(x) = \dfrac{x^3}{x^2+2|x|+1}$.

Then, $f(-x) = \dfrac{(-x)^3}{(-x)^2+2|-x|+1} = -\dfrac{x^3}{x^2+2|x|+1} = -f(x)$.

$\therefore \quad I_1 = 0$

Now, $I_2 = \displaystyle\int_{-1}^{1} \frac{|x|+1}{x^2+2|x|+1}\,dx.$

Assume $f(x) = \dfrac{|x|+1}{x^2+2|x|+1}.$

Then, $f(-x) = \dfrac{|-x|+1}{(-x)^2+2|-x|+1} = \dfrac{|x|+1}{x^2+2|x|+1} = f(x).$

$$\begin{aligned}
\therefore\quad I_2 &= 2\int_{0}^{1} \frac{|x|+1}{x^2+2|x|+1}\,dx \\
&= 2\int_{0}^{1} \frac{x+1}{x^2+2x+1}\,dx \\
&= 2\int_{0}^{1} \frac{x+1}{(x+1)^2}\,dx \\
&= 2\int_{0}^{1} \frac{1}{x+1}\,dx \\
&= 2[\log|x+1|]_0^1 \\
&= 2[\log 2 - 0] \\
&= 2\log 2.
\end{aligned}$$

Putting values of I_1 and I_2 in (1), we get

$$I = 0 + 2\log 2 = 2\log 2.$$

■

27. Given $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}.$

Then, $|A| = \begin{vmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{vmatrix} = 2(-4+4) + 3(-6+4) + 5(3-2) = -1 \neq 0.$

$\therefore A$ is non-singular and hence invertible.

Let A_{ij} denote cofactor of a_{ij} in $A = [a_{ij}]$, then

$A_{11} = (-1)^{1+1}\begin{vmatrix} 2 & -4 \\ 1 & -2 \end{vmatrix} = 0,$ $A_{12} = (-1)^{1+2}\begin{vmatrix} 3 & -4 \\ 1 & -2 \end{vmatrix} = 2,$

$A_{13} = (-1)^{1+3}\begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} = 1,$ $A_{21} = (-1)^{2+1}\begin{vmatrix} -3 & 5 \\ 1 & -2 \end{vmatrix} = -1,$

$A_{22} = (-1)^{2+2}\begin{vmatrix} 2 & 5 \\ 1 & -2 \end{vmatrix} = -9,$ $A_{23} = (-1)^{2+3}\begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} = -5,$

$A_{31} = (-1)^{3+1}\begin{vmatrix} -3 & 5 \\ 2 & -4 \end{vmatrix} = 2,$ $A_{32} = (-1)^{3+2}\begin{vmatrix} 2 & 5 \\ 3 & -4 \end{vmatrix} = 23,$

$A_{33} = (-1)^{3+3}\begin{vmatrix} 2 & -3 \\ 3 & 2 \end{vmatrix} = 13.$

$$\therefore \text{adj } A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}$$

$$\text{and } A^{-1} = \frac{1}{|A|}(\text{adj } A) = -\begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}.$$

Now, given system of equations is

$$\begin{aligned} 2x - 3y + 5z &= 11 \\ 3x + 2y - 4z &= -5 \\ x + y - 2z &= -3 \end{aligned}$$

which can be written in matrix form as

$$\begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$$

i.e., $AX = B,$

$$\text{where } A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}.$$

Since, $|A| = -1 \neq 0$.

$\therefore$ Given system of equations is consistent and has unique solution given by $X = A^{-1}B$.

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = X = A^{-1}B = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix} \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

$\Rightarrow \quad x = 1, y = 2, z = 3$

Hence, the required solution is $x = 1$, $y = 2$, $z = 3$.

OR

$$\text{Let} \quad \Delta = \begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix}$$

$$= \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ y & y^2 & y^3 \\ z & z^2 & z^3 \end{vmatrix}$$

$$= \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + xyz \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

[On taking common x from R_1, y from R_2 and z from R_3 in second determinant]

$$= \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} - xyz \begin{vmatrix} x & 1 & x^2 \\ y & 1 & y^2 \\ z & 1 & z^2 \end{vmatrix}$$

[On applying $C_1 \leftrightarrow C_2$ in second determinant]

$$= \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + xyz \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} \quad \begin{bmatrix} \text{On applying } C_2 \leftrightarrow C_3 \\ \text{in second determinant} \end{bmatrix}$$

$$= (1+xyz) \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix}$$

$$= (1+xyz) \begin{vmatrix} x & x^2 & 1 \\ y-x & y^2-x^2 & 0 \\ z-x & z^2-x^2 & 0 \end{vmatrix} \quad \begin{bmatrix} \text{On applying} \\ R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{bmatrix}$$

$$= (1+xyz)(y-x)(z-x) \begin{vmatrix} x & x^2 & 1 \\ 1 & y+x & 0 \\ 1 & z+x & 0 \end{vmatrix} \quad \begin{bmatrix} \text{On taking common} \\ (y-x) \text{ from } R_2 \\ \text{and } (z-x) \text{ from } R_3 \end{bmatrix}$$

$$= (1+xyz)(y-x)(z-x)[1\{(z+x)-(y+x)\}-0+0] \quad \left[\text{On expanding along } C_3\right]$$

$$\therefore \qquad \Delta = (1+xyz)(y-x)(z-x)(z-y)$$

Given that $\qquad \Delta = 0$

$$\Rightarrow \qquad (1+xyz)(y-x)(z-x)(z-y) = 0$$

$$\Rightarrow \qquad 1 + xyz = 0. \quad \begin{bmatrix} \because y-x \neq 0, z-x \neq 0, z-y \neq 0, \\ \text{as } x \neq y \neq z \text{ (given)} \end{bmatrix}$$ ■

28. Let the number of large boxes $= x$

and the number of small boxes $= y$

Let total profit $=$ ₹ Z

We can represent the given L.P.P. in the following tabular form:

	Large Boxes	**Small Boxes**	**Requirement**
Profit (₹)	$3x$	$2y$	Maximise
Cardboard used (sq m)	$4x$	$3y$	At most 60
Number of large boxes	x	–	At least 3
Number of small boxes	–	y	At most $2x$

Hence, given L.P.P. is, Maximise $Z = 3x + 2y$

subject to the constraints:

$$4x + 3y \leq 60, \quad x \geq 3, \quad y \leq 2x, \quad x \geq 0, \quad y \geq 0$$

We consider the following equations:

$4x + 3y = 60$

x	0	15
y	20	0

$y = 2x$

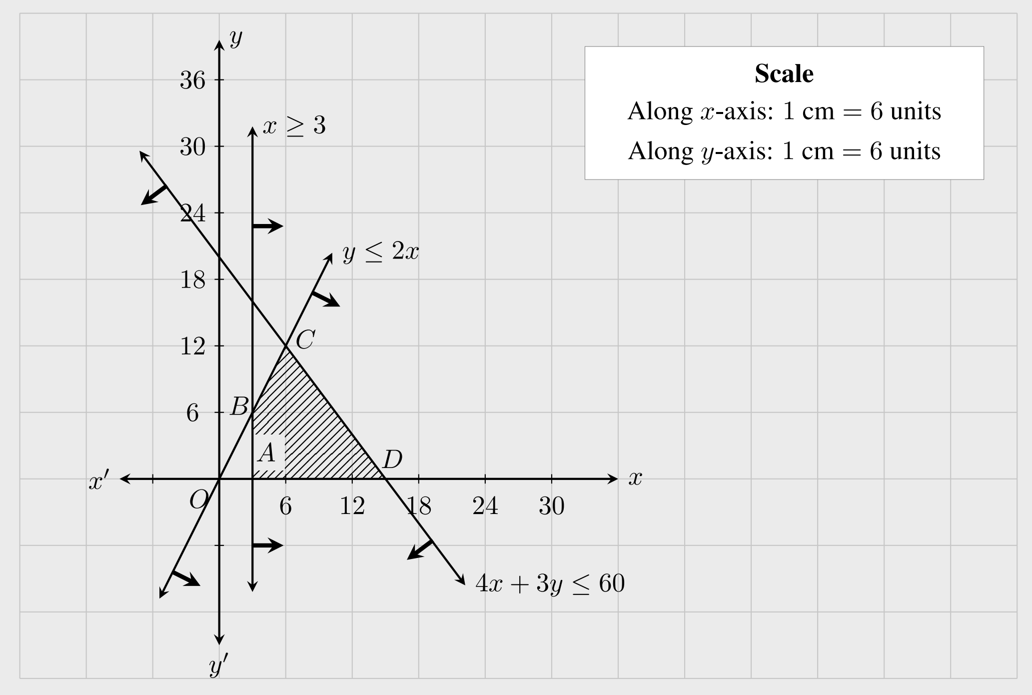
i.e $2x - y = 0$

x	0	3
y	0	6

$x = 3$,

$x = 0, y = 0$

The feasible region of L.P.P. is bounded, as shown shaded in the graph.



Corner Points	Value of $Z (Z = 3x + 2y)$
$A(3, 0)$	9
$B(3, 6)$	21
$C(6, 12)$	42
$D(15, 0)$	45

Since, the feasible region is bounded and 45 is the maximum value of Z at corner points.
$\therefore$ 45 is the maximum value of Z in the feasible region at $x = 15$, $y = 0$.
Hence, number of large boxes = 15, number of small boxes = 0
and maximum profit = ₹ 45.

OR

Let the number of cakes of I kind $= x$

and the number of cakes of II kind $= y$

Let total number of cakes $= Z$

We can represent the given L.P.P. in the following tabular form:

	I Kind	II Kind	Requirement
Number of cakes	x	y	Maximise
Flour (g)	$200x$	$100y$	At most (5×1000)
Fat (g)	$25x$	$50y$	At most (1×1000)

Hence, given L.P.P. is, Maximise $Z = x + y$

subject to the constraints:

$$200x + 100y \leq 5000, \quad 25x + 50y \leq 1000, \quad x \geq 0, \quad y \geq 0$$

We consider the following equations:

$200x + 100y = 5000$

i.e., $2x + y = 50$

x	0	25
y	50	0

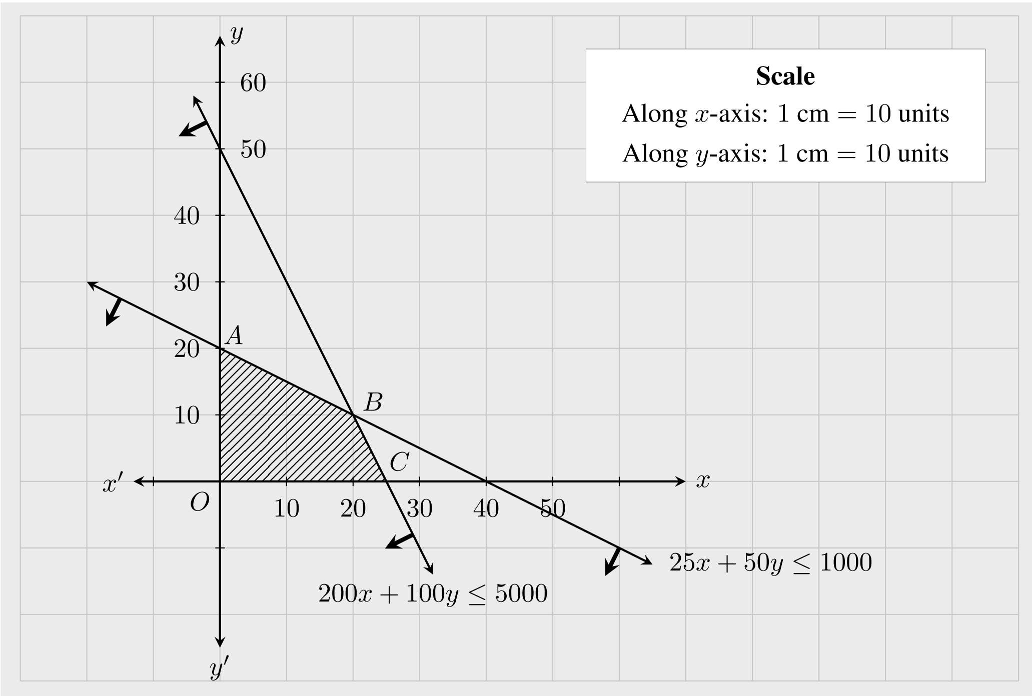
$25x + 50y = 1000$

i.e., $x + 2y = 40$

x	0	40
y	20	0

$x = 0, \; y = 0$

The feasible region of L.P.P. is bounded, as shown shaded in the graph.



Corner Points	Value of $Z (Z = x + y)$
$A(0, 20)$	20
$B(20, 10)$	30
$C(25, 0)$	25
$O(0, 0)$	0

Since, the feasible region is bounded and 30 is the maximum value of Z at corner points.

$\therefore$ 30 is the maximum value of Z in the feasible region at $x = 20$, $y = 10$.

Hence, number of cakes of I kind = 20, number of cakes of II kind = 10

and maximum number of cakes = 30. ■

29. Given curves are $y = x^2$ and $y = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x \leq 0 \end{cases}$.

$\therefore$ Given curves are

(i) $y = x^2$ (ii) $y = x$, if $x \geq 0$ (iii) $y = -x$, if $x \leq 0$

To find points of intersection

From (i) and (ii), we have

$$x^2 = x$$

$$\Rightarrow \quad x(x-1) = 0$$

$$\Rightarrow \quad x = 0, 1$$

$\therefore$ Curves (i) and (ii) intersect each other at $(0, 0)$ and $(1, 1)$.

From (i) and (iii), we have

$$x^2 = -x$$

$$\Rightarrow \quad x(x+1) = 0$$

$$\Rightarrow \quad x = 0, -1$$

$\therefore$ Curves (i) and (iii) intersect each other at $(0, 0)$ and $(-1, 1)$.

To plot the required region

$y = x^2$

x	0	1	-1	$\frac{1}{2}$	$-\frac{1}{2}$
y	0	1	1	$\frac{1}{4}$	$\frac{1}{4}$

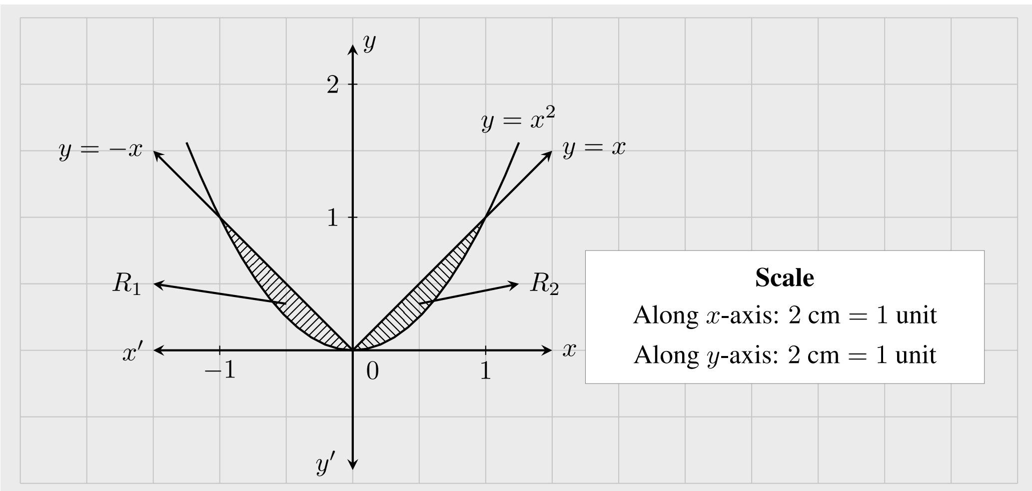
$y = x$ (if $x \geq 0$)

x	0	1
y	0	1

$y = -x$ (if $x \leq 0$)

x	0	-1
y	0	1

The required region is shown shaded in the graph.



To find the area of required region

$A =$ Area of region R_1 + Area of region R_2

$$= \int_{-1}^{0} [-x - x^2]\, dx + \int_{0}^{1} [x - x^2]\, dx$$

$$= \left[-\frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{0} + \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{0}^{1}$$

$$= \left[(0-0) - \left(-\frac{1}{2} + \frac{1}{3}\right)\right] + \left[\left(\frac{1}{2} - \frac{1}{3}\right) - (0-0)\right]$$

$$= \frac{1}{6} + \frac{1}{6} = \frac{1}{3}.$$

Hence, required area $= \frac{1}{3}$ sq. unit. ■

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Unsolved Practice Paper – 1

Time: 3 Hours **Max. Marks: 100**

General Instructions

(i) **All** questions are compulsory.
(ii) Please check that this question paper consists of **29** questions.
(iii) Questions **1** to **4** in Section A are Very Short Answer Type Questions carrying **1** mark each.
(iv) Questions **5** to **12** in Section B are Short Answer I Type Questions carrying **2** marks each.
(v) Questions **13** to **23** in Section C are Long Answer I Type Questions carrying **4** marks each.
(vi) Questions **24** to **29** in Section D are Long Answer II Type Questions carrying **6** marks each.
(vii) Please write down the serial number of the question before attempting it.

Section A

Question numbers **1** *to* **4** *carry* **1** *mark each.*

1. If $\vec{a} = x\hat{i} + 2\hat{j} - z\hat{k}$ and $\vec{b} = 3\hat{i} - y\hat{j} + \hat{k}$ are two equal vectors, then write the value of $x + y + z$.

2. Find $g \circ f$ and $f \circ g$, when $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are defined by $f(x) = 8x^3$ and $g(x) = x^{1/3}$.

3. Find the principal value of $\cos^{-1}\left(-\dfrac{\sqrt{3}}{2}\right)$.

4. If a matrix has 18 elements, what are the possible orders it can have? What if it has 5 elements?

Section B

Question numbers **5** *to* **12** *carry* **2** *marks each.*

5. A and B are two candidates seeking admission in a college. The probability that A is selected is 0.7 and the probability that exactly one of them is selected is 0.6. Find the probability that B is selected.

6. Evaluate: $\displaystyle\int \frac{x\,e^x}{(x+1)^2}\,dx$

7. Let $A = \{1, 2, 3, 4\}$ and $B = \{5, 6, 7, 8\}$. Let $f : A \to B$ be defined by

$$f = \{(1, 5), (2, 6), (3, 6), (4, 7)\}.$$

Show that f is neither one-one nor onto.

8. If $M(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, then show that $M(x)M(y) = M(x + y)$.

9. Using differentials, find the approximate value of $\cot^{-1}(1.004)$.

10. Find the points on the curve $y = \cos x - 1$ in $[0, 2\pi]$, where the tangent is parallel to x-axis.

11. Prove that $\begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix} = 2(x+y+z)^3$.

12. Evaluate: $\displaystyle\int \left[1 + 2\tan x\,(\tan x + \sec x)\right]^{1/2} dx$

Section C

Question numbers **13** *to* **23** *carry* **4** *marks each.*

13. Two cards are drawn successively without replacement from a well shuffled deck of cards. Find the mean, variance and standard deviation of the random variable X, where X is the number of aces.

OR

There are three identical boxes I, II and III each containing two coins. In Box I, both coins are gold coins; in Box II, both are silver coins and in Box III, there is one gold and one silver coin. A person chooses a box at random and takes out a coin. If the coin is of gold, what is the probability that the other coin in the box is also of gold?

14. If $\hat{a}$ and $\hat{b}$ are unit vectors inclined at an angle θ, then prove that

(i) $|\hat{a} - \hat{b}| = 2\sin\dfrac{\theta}{2}$. (ii) $|\hat{a} + \hat{b}| = 2\cos\dfrac{\theta}{2}$.

15. Show that the differential equation $x\,\dfrac{dy}{dx}\,\sin\left(\dfrac{y}{x}\right) + x - y\,\sin\left(\dfrac{y}{x}\right) = 0$ is homogeneous. Find the particular solution of this differential equation, given that $x = 1$ when $y = \dfrac{\pi}{2}$.

16. If $A = \begin{bmatrix} 3 & 2 \\ 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ -4 & 0 \end{bmatrix}$, then find $(AB)^{-1}$.

17. A ladder 5 m long is leaning against a wall. The bottom of the ladder is pulled along the ground away from the wall at the rate of 2 cm/s. How fast is its height on the wall decreasing when the foot of the ladder is 4 m away from the wall?

18. If $x = a(\theta - \sin\theta)$ and $y = a(1 + \cos\theta)$, then prove that $\dfrac{d^2y}{dx^2} = \dfrac{1}{a(1-\cos\theta)^2}$.

19. Evaluate: $\displaystyle\int \frac{1-x^2}{x(1-2x)}\,dx$.

20. Prove that $\begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix} = 3(a+b+c)(ab+bc+ca)$.

21. A and B appeared for an interview for two vacancies. The probability of A's selection is $\dfrac{1}{5}$ and that of B's selection is $\dfrac{1}{3}$. Find the probability that

(i) both of them will be selected. (ii) none of them will be selected.

(iii) only one of them will be selected. (iv) at least one will be selected.

Name two qualities that a person should possess while appearing for an interview.

22. Evaluate: $\displaystyle\int_{\pi/6}^{\pi/3} \frac{1}{1+\sqrt{\cot x}}\,dx$

OR

Evaluate: $\displaystyle\int_{-1}^{3/2} |x\,\sin(\pi x)|\,dx$

23. Find the values of a and b such that $f(x) = \begin{cases} 5 & \text{if } x \le 2 \\ ax + b & \text{if } 2 < x < 10 \\ 21 & \text{if } x \ge 10 \end{cases}$ is a continuous function.

OR

Examine the differentiability of function f defined by $f(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{cases}$ at $x = 0$.

Section D

Question numbers **24** *to* **29** *carry* **6** *marks each.*

24. Write the nature of the lines

$$\vec{r} = (4\hat{i} - \hat{j} + 0\hat{k}) + \lambda(\hat{i} + 2\hat{j} - 3\hat{k}) \text{ and } \vec{r} = (\hat{i} - \hat{j} + 2\hat{k}) + \mu(2\hat{i} + 4\hat{j} - 5\hat{k}).$$

Also, find the shortest distance between them.

OR

Find the equation of the plane passing through the line of intersection of the planes having equations $x + 2y + 3z - 5 = 0$ and $3x - 2y - z + 1 = 0$ and cutting off equal intercepts on the x and z-axes.

25. Let $A = \{1, 2, 3, \ldots, 9\}$ and R be the relation in $A \times A$ defined by $(a, b)\ R\ (c, d)$ if $a + d = b + c$ for (a, b), (c, d) in $A \times A$. Prove that R is an equivalence relation. Also obtain the equivalence class $[(2, 5)]$.

OR

Solve the equation for x: $\sin^{-1}\dfrac{5}{x} + \sin^{-1}\dfrac{12}{x} = \dfrac{\pi}{2}, x \ne 0.$

26. Find the area bounded by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ and the ordinates $x = 0$, $x = ae$, where $b^2 = a^2(1 - e^2)$ and $e < 1$.

27. A company produces two types of items, P and Q. Manufacturing of both items requires the metals gold and copper. Each unit of item P requires 3 grams of gold and 1 gm of copper while that of item Q requires 1 gm of gold and 2 grams of copper. The company has 9 grams of gold and 8 grams of copper in its store. If each unit of item P makes a profit of ₹ 50 and each unit of item Q makes a profit of ₹ 60, determine the number of units of each item that the company should produce to maximise profit. What is the maximum profit?

28. Show that the semi-vertical angle of a cone of maximum volume and given slant height is given by $\tan^{-1}\sqrt{2}$.

OR

An isosceles triangle of vertical angle 2θ is inscribed in a circle of radius a. Show that the area of triangle is maximum when $\theta = \dfrac{\pi}{6}$.

29. Find the coordinates of the point where the line through $(5, 1, 6)$ and $(3, 4, 1)$ crosses the xz-plane.

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Unsolved Practice Paper – 2

Time: 3 Hours **Max. Marks: 100**

General Instructions

(i) **All** questions are compulsory.

(ii) Please check that this question paper consists of **29** questions.

(iii) Questions **1** to **4** in Section A are Very Short Answer Type Questions carrying **1** mark each.

(iv) Questions **5** to **12** in Section B are Short Answer I Type Questions carrying **2** marks each.

(v) Questions **13** to **23** in Section C are Long Answer I Type Questions carrying **4** marks each.

(vi) Questions **24** to **29** in Section D are Long Answer II Type Questions carrying **6** marks each.

(vii) Please write down the serial number of the question before attempting it.

Section A

Question numbers **1** *to* **4** *carry* **1** *mark each.*

1. Prove that: $\sin^{-1}\dfrac{8}{17} + \sin^{-1}\dfrac{3}{5} = \cos^{-1}\dfrac{36}{85}$.

2. Let $*$ be an operation defined on $\mathbb{N}$ as $a * b = \dfrac{a+b}{2}$ for $a, b \in \mathbb{N}$. Is $*$ a binary operation?

3. If $\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & -2 \\ 7 & 3 \end{vmatrix}$, write the value of x.

4. Find the angle between vectors $\vec{a}$ and $\vec{b}$ with magnitudes 1 and 2 respectively and $|\vec{a} \times \vec{b}| = \sqrt{3}$.

Section B

Question numbers **5** *to* **12** *carry* **2** *marks each.*

5. Find the intervals on which the function $f(x) = \sin^4 x + \cos^4 x$ on $\left[0, \dfrac{\pi}{2}\right]$ is (I) strictly increasing or strictly decreasing, (II) increasing or decreasing.

6. Using properties of determinants, evaluate: $\begin{vmatrix} 9! & 10! & 11! \\ 10! & 11! & 12! \\ 11! & 12! & 13! \end{vmatrix}$.

7. Evaluate: $\displaystyle\int_{-\pi/2}^{\pi/2} (\sin|x| + \cos|x|)\, dx$.

8. For any three vectors $\vec{a}, \vec{b}, \vec{c}$, prove: $[\, \vec{a} - \vec{b} \;\; \vec{b} - \vec{c} \;\; \vec{c} - \vec{a}\,] = 0$.

9. It is given that Rolle's theorem holds good for the function $f(x) = x^3 + ax^2 + bx, x \in [1, 2]$ at the point $x = \dfrac{4}{3}$. Find the values of a and b.

10. Prove that: $\tan\left[\dfrac{1}{2}\left\{\sin^{-1}\left(\dfrac{2x}{1+x^2}\right) + \cos^{-1}\left(\dfrac{1-y^2}{1+y^2}\right)\right\}\right] = \dfrac{x+y}{1-xy}$; $|x| < 1, y > 0, xy < 1$.

11. Differentiate w.r.t. x: $\tan^{-1}\left(\dfrac{a\cos x - b\sin x}{b\cos x + a\sin x}\right)$.

12. Given $A = \begin{bmatrix} 2 & 4 & 0 \\ 3 & 9 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 4 \\ 2 & 8 \\ 1 & 3 \end{bmatrix}$. Is $(AB)^T = B^T A^T$?

Section C

Question numbers **13** *to* **23** *carry* **4** *marks each.*

13. Evaluate: $\displaystyle\int \frac{\sqrt{x^2+1}\left[\log\left(x^2+1\right)-2\log x\right]}{x^4}\,dx.$

14. There are two families F_1 and F_2. There are 6 men, 5 women and 2 children in family F_1 and 3 men, 3 women and 4 children in family F_2. The recommended daily requirement for proteins is—Man: 70 g, Woman: 50 g, Child: 30 g and for carbohydrates is—Man: 500 g, Woman: 400 g, Child: 300 g. Using matrix multiplication, calculate the total requirement of proteins and carbohydrates for each of the two families. **What other nutrients should be included in the diet for a healthy living? Mention any three.**

15. Discuss the continuity of $f(x)=\begin{cases}\dfrac{1-\cos 2x}{x^2} & \text{if } x\neq 0\\ 5 & \text{if } x=0\end{cases}$ at $x=0$.

OR

Find the relation between a and b so that $f(x)=\begin{cases}ax+1 & \text{if } x\leq 3\\ bx+3 & \text{if } x>3\end{cases}$ is continuous at $x=3$.

16. The probability that A hits a target is $\dfrac{1}{3}$ and the probability that B hits it is $\dfrac{2}{5}$. If each one of A and B shoots at the target, what is the probability that

(i) the target is hit? (ii) exactly one of them hits the target?

OR

There are two bags. One bag contains six green and three red balls. The second bag contains five green and four red balls. One ball is transferred from the first bag to the second bag. The one ball is drawn from the second bag. Find the probability that it is a red ball.

17. Find the equation of the plane passing through the points $(1,1,0)$, $(1,2,1)$ and $(-2,2,-1)$.

18. Evaluate $\displaystyle\int_1^4 \left(x^2-x\right)\,dx$ as the limit of sum.

19. Find the particular solution of the differential equation $\dfrac{dy}{dx}+y\cot x=4x\operatorname{cosec} x,\ (x\neq 0)$, given that $y=0$ when $x=\dfrac{\pi}{2}$.

20. Find a unit vector perpendicular to each of the vectors $(\vec{a}+\vec{b})$ and $(\vec{a}-\vec{b})$, where $\vec{a}=3\hat{i}+2\hat{j}+2\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}-2\hat{k}$.

OR

If $\vec{a},\vec{b},\vec{c}$ are three vectors such that $\vec{a}\times\vec{b}=\vec{c},\ \vec{b}\times\vec{c}=\vec{a}$, prove that $\vec{a},\vec{b},\vec{c}$ are mutually perpendicular to each other with $|\vec{b}|=1$ and $|\vec{a}|=|\vec{c}|$.

21. If $y=Ae^{mx}+Be^{nx}$, then prove that $y_2-(m+n)y_1+mny=0$.

22. Without expanding, prove that $\begin{vmatrix} b+c & q+r & y+z\\ c+a & r+p & z+x\\ a+b & p+q & x+y\end{vmatrix}=2\begin{vmatrix} a & p & x\\ b & q & y\\ c & r & z\end{vmatrix}$.

23. Find the equation of the tangent to the curve $x = \sin 3t$, $y = \cos 2t$ at $t = \frac{\pi}{4}$.

Section D

Question numbers **24** *to* **29** *carry* **6** *marks each.*

24. A fruit grower can use two types of fertilisers in his garden, brand P and brand Q. The amounts (in kg) of nitrogen, phosphoric acid, potash and chlorine in a bag of each brand are given in the table. Tests indicate that the garden needs at least 240 kg of phosphoric acid, at least 270 kg of potash and at most 310 kg of chlorine. If the grower wants to minimise the amount of nitrogen added to the garden, how many bags of each brand should be used? What is the minimum amount of nitrogen added in the garden? Formulate the above L.P.P. mathematically and then solve it graphically.

	Brand P	**Brand Q**
Nitrogen	3	3.5
Phosphoric acid	1	2
Potash	3	1.5
Chlorine	1.5	2

25. Find the coordinates of the foot of perpendicular from the point $(2, 3, 7)$ to the plane $3x - y - z = 7$. Also, find the length of the perpendicular.

OR

Show that the lines $\vec{r} = (3\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(\hat{i} + 2\hat{j} + 2\hat{k})$ and $\vec{r} = (5\hat{i} - 2\hat{j}) + \mu(3\hat{i} + 2\hat{j} + 6\hat{k})$ are intersecting. Hence, find their point of intersection.

26. In a group of 50 students in a camp, 30 are well trained in first aid techniques while the remaining are well trained in hospitality but not in first aid. Two scouts are selected at random from the group. Find the probability distribution of number of selected scouts who are well trained in first aid. Find the mean of the distribution also.

OR

In an examination, an examinee either guesses or copies or knows the answer of MCQs with four choices. The probability that he makes a guess is $\frac{1}{3}$ and the probability that he copies answer is $\frac{1}{6}$. The probability that his answer is correct, given that he copied it, is $\frac{1}{8}$. Find the probability that he copies the answer to question, given that he correctly answered it.

27. Consider the non-empty set consisting of children in a family and a relation R defined as $a\ R\ b$ if a is brother of b. Determine whether R is reflexive, symmetric and transitive.

28. The sum of the perimeters of a square and a circle is k, where k is some constant. Show that the sum of their areas is least when the side of the square is equal to the diameter of circle.

29. Find the area enclosed by the parabola $4y = 3x^2$ and the line $2y = 3x + 12$.

OR

Show that the function f defined by $f(x) = |1 - x + |x||$, where x is any real number, is a continuous function.

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Unsolved Practice Paper – 3

Time: 3 Hours **Max. Marks: 100**

General Instructions

(i) **All** questions are compulsory.

(ii) Please check that this question paper consists of **29** questions.

(iii) Questions **1** to **4** in Section A are Very Short Answer Type Questions carrying **1** mark each.

(iv) Questions **5** to **12** in Section B are Short Answer I Type Questions carrying **2** marks each.

(v) Questions **13** to **23** in Section C are Long Answer I Type Questions carrying **4** marks each.

(vi) Questions **24** to **29** in Section D are Long Answer II Type Questions carrying **6** marks each.

(vii) Please write down the serial number of the question before attempting it.

Section A

Question numbers **1** *to* **4** *carry* **1** *mark each.*

1. Given two sets $S = \{a, b, c\}$ and $T = \{1, 2, 3\}$. Find f^{-1} of the function $f : S \to T$ defined by $f = \{(a, 2), (b, 1), (c, 1)\}$, if it exists.

2. Find the value of λ such that the vectors $\vec{a} = 2\hat{i} + \lambda\hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$ are orthogonal.

3. Evaluate: $\tan\left[2\tan^{-1}\dfrac{1}{5}\right]$.

4. If $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, then show that $A^2 = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix}$.

Section B

Question numbers **5** *to* **12** *carry* **2** *marks each.*

5. For what value of a the vectors $2\hat{i} - 3\hat{j} + 4\hat{k}$ and $a\hat{i} + 6\hat{j} - 8\hat{k}$ are collinear?

6. Relation R on the set $A = \{1, 2, 3, \ldots, 14\}$ is defined as $R = \{(x, y) : 3x - y = 0\}$. Determine whether the given relation is reflexive, symmetric and transitive.

7. If $y = \sin^{-1}\left(\dfrac{1}{\sqrt{x+1}}\right)$, then find $\dfrac{dy}{dx}$.

8. Evaluate: $\displaystyle\int \frac{x}{\sqrt{x}+1}\, dx$.

9. Evaluate: $\displaystyle\int \frac{1}{\sin(x-a)\ \cos(x-b)}\, dx$

10. If $a_1,\ a_2,\ a_3,\ \ldots,\ a_r$ are in G.P., then prove that the determinant $\begin{vmatrix} a_{r+1} & a_{r+5} & a_{r+9} \\ a_{r+7} & a_{r+11} & a_{r+15} \\ a_{r+11} & a_{r+17} & a_{r+21} \end{vmatrix}$ is independent of r.

11. Find the intervals on which the function $f(x) = 5x^{3/2} - 3x^{5/2}$ on $(0, \infty)$ is (I) strictly increasing or strictly decreasing, (II) increasing or decreasing.

12. Find the point on the curve $y^2 = x$, where the tangent makes an angle of $\dfrac{\pi}{4}$ with the x-axis.

Section C

Question numbers **13** *to* **23** *carry* **4** *marks each.*

13. Evaluate: $\displaystyle\int_0^{\pi/2} \frac{x\ \sin x\ \cos x}{\sin^4 x + \cos^4 x}\, dx$

14. Solve the differential equation $(x^2 - yx^2)\, dy + (y^2 + x^2y^2)\, dx = 0$, given that $y = 1$ when $x = 1$.

15. Differentiate w.r.t. x: $\sin^{-1}\left(x\sqrt{1-x^4} - x^2\sqrt{1-x^2}\right)$.

OR

If $e^y(1+x) = 1$, then prove that $y_2 = (y_1)^2$.

16. Prove that: $\begin{vmatrix} 1 & a^2+bc & a^3 \\ 1 & b^2+ca & b^3 \\ 1 & c^2+ab & c^3 \end{vmatrix} = -(a-b)(b-c)(c-a)(a^2+b^2+c^2).$

OR

Prove that: $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c).$

17. If with reference to the right-handed system of mutually perpendicular unit vectors $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$, we have, $\vec{\alpha} = 3\hat{\imath} - \hat{\jmath}$, $\vec{\beta} = 2\hat{\imath} + \hat{\jmath} - 3\hat{k}$, then express $\vec{\beta}$ in the form $\vec{\beta} = \vec{\beta}_1 + \vec{\beta}_2$, where $\vec{\beta}_1$ is parallel to $\vec{\alpha}$ and $\vec{\beta}_2$ is perpendicular to $\vec{\alpha}$.

18. Evaluate: $\displaystyle\int \frac{5x-2}{1+2x+3x^2}\, dx.$

19. If the function $f(x) = \begin{cases} 3ax+b & \text{if } x > 1 \\ 11 & \text{if } x = 1 \\ 5ax-2b & \text{if } x < 1 \end{cases}$ is continuous at $x = 1$, find the values of a and b.

20. Show that the lines $\dfrac{5-x}{-4} = \dfrac{y-7}{4} = \dfrac{z+3}{-5}$ and $\dfrac{x-8}{7} = \dfrac{2y-8}{2} = \dfrac{z-5}{3}$ are coplanar.

21. A car manufacturing factory has two plants, X and Y. Plant X manufactures 70% of cars and plant Y manufactures 30%. 80% of the cars at plant X and 90% of the cars at plant Y are rated of standard quality. A car is chosen at random and is found to be of standard quality. What is the probability that it has come from plant X?

OR

A speaks truth in 60% of the cases, while B in 90% of the cases. In what per cent of cases are they likely to contradict each other in stating the same fact?

22. In a competition, a brave child tries to inflate a huge spherical balloon bearing slogans against child labour at the rate of 900 cubic cm of gas per second. Find the rate at which the radius of the balloon is increasing when its radius is 15 cm. **Write any three values / life skills reflected in this question.**

23. Show that $f(x) = 2x + \cot^{-1}x + \log\left(\sqrt{1+x^2} - x\right)$ is increasing on $\mathbb{R}$.

Section D

Question numbers **24** *to* **29** *carry* **6** *marks each.*

24. From a lot of 15 bulbs, which include 5 defective bulbs, a sample of 4 bulbs is drawn one by one with replacement. Find the probability distribution of the number of defective bulbs. Hence, find the mean of the distribution.

OR

Suppose the reliability of an HIV test is specified as follows: Among people having HIV, 90% of the tests detect the disease but 10% fail to do so. Among people not having HIV, 99% of the tests show HIV $-$ve but 1% are diagnosed as HIV $+$ve. From a large population of which only 0.1% have HIV, one person is selected at random for an HIV test, and the pathologist reports him/her as HIV $+$ve. What is the probability that the person actually has HIV?

25. Sketch the graph of the curve $y = |x+3|$ and evaluate $\displaystyle\int_{-6}^{0} |x+3|\, dx$.

26. The sum of three numbers is 6. If we multiply the third number by 3 and add second number to it, we get 11. By adding the first and third numbers, we get double of the second number. Represent it algebraically and find the numbers using matrix method.

27. Show that the plane whose vector equation is $\vec{r} \cdot (\hat{i} + 2\hat{j} - \hat{k}) = 6$ contains the line whose vector equation is $\vec{r} = (4\hat{i} + \hat{j}) + \lambda(2\hat{i} + \hat{j} + 4\hat{k})$.

OR

Find the coordinates of the foot of the perpendicular and the length of the perpendicular drawn from the point $P(5, 4, 2)$ to the line $\vec{r} = -\hat{i} + 3\hat{j} + \hat{k} + \lambda(2\hat{i} + 3\hat{j} - \hat{k})$. Also, find the image of P in this line.

28. Show that the given relation R is defined on the set $A = \{x \in \mathbb{Z} : 0 \leq x \leq 12\}$, given by $R = \{(a, b) : |a - b| \text{ is multiple of } 4\}$, is an equivalence relation. Write the set of all elements related to 1.

OR

Solve: $\sin^{-1}x + \sin^{-1}2x = \dfrac{\pi}{3}$.

29. Two tailors A and B earn ₹ 150 and ₹ 200 per day respectively. The tailor A can stitch 6 shirts and 4 trousers a day, while B can stitch 10 shirts and 4 trousers per day. How many days should each work to produce at least 60 shirts and 32 trousers at a minimum labour cost? Formulate the above L.P.P. mathematically and then solve it graphically.

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Unsolved Practice Paper – 4

Time: 3 Hours **Max. Marks: 100**

General Instructions

(i) **All** questions are compulsory.

(ii) Please check that this question paper consists of **29** questions.

(iii) Questions **1** to **4** in Section A are Very Short Answer Type Questions carrying **1** mark each.

(iv) Questions **5** to **12** in Section B are Short Answer I Type Questions carrying **2** marks each.

(v) Questions **13** to **23** in Section C are Long Answer I Type Questions carrying **4** marks each.

(vi) Questions **24** to **29** in Section D are Long Answer II Type Questions carrying **6** marks each.

(vii) Please write down the serial number of the question before attempting it.

Section A

Question numbers **1** *to* **4** *carry* **1** *mark each.*

1. Evaluate: $2\sec^{-1}2+\sin^{-1}\left(\frac{1}{2}\right)$.

2. If A is a square matrix of order 3×3 such that $|A|=3$, then find $|A(\text{adj } A)|$.

3. Find the angle between vectors $\vec{a}$ and $\vec{b}$ with magnitudes $\sqrt{3}$ and 2 respectively having $\vec{a}\cdot\vec{b}=\sqrt{6}$.

4. Evaluate: $\tan^{-1}\left(\tan\frac{7\pi}{6}\right)$.

Section B

Question numbers **5** *to* **12** *carry* **2** *marks each.*

5. Evaluate: $\displaystyle\int_{1/3}^{1}\frac{\left(x-x^3\right)^{1/3}}{x^4}\,dx$.

6. In a school, there are 1000 students, out of which 430 are girls. It is known that out of 430, 10% of the girls study in class XII. What is the probability that a student chosen randomly studies in class XII, given that the chosen student is a girl?

7. Prove, using properties of determinants: $\begin{vmatrix}-a^2 & ab & ac\\ ab & -b^2 & bc\\ ac & bc & -c^2\end{vmatrix}=4a^2b^2c^2$.

8. Determine whether the binary operation $*$ on $\mathbb{R}$ defined by $a*b=ab+1$ is commutative and associative.

9. Find the area of the parallelogram whose diagonals are $2\hat{i}-\hat{j}+\hat{k}$ and $\hat{i}+3\hat{j}-\hat{k}$.

10. The two equal sides of an isosceles triangle with fixed base 'b' are decreasing at the rate of 3 cm/s. How fast is the area decreasing when the two equal sides are equal to the base?

11. Find a point on the curve $y=(x-3)^2$, where the tangent is parallel to the line joining $(4,-1)$ and $(5,0)$.

12. Find the Cartesian equation of the line which passes through the point $(-2,4,-5)$ and is parallel to the line $\frac{x+3}{3}=\frac{4-y}{5}=\frac{z+8}{6}$.

Section C

Question numbers **13** *to* **23** *carry* **4** *marks each.*

13. Prove that the lines $x = py + q$, $z = ry + s$ and $x = p'y + q'$, $z = r'y + s'$ are perpendicular if $pp' + rr' + 1 = 0$.

14. Show that the curves $x = y^2$ and $xy = k$ cut at right angles if $8k^2 = 1$.

15. Suppose 5% of men and 0.25% of women have grey hair. A grey-haired person is selected at random. What is the probability of this person being male? Assume that there are equal number of males and females.

OR

A purse contains 4 silver and 5 copper coins. Another purse contains 3 silver and 7 copper coins. If a coin is taken out at random from one of the purses, what is the probability that it is a copper coin?

16. Show that the points $(0, -1, 0)$, $(1, 1, 1)$, $(3, 3, 0)$ and $(0, 1, 3)$ are coplanar. Also, find the plane containing them.

17. Evaluate: $\displaystyle\int \frac{1}{\sin^4 x + \sin^2 x\ \cos^2 x + \cos^4 x}\, dx.$

18. If $x^y = e^{x-y}$, then prove that $\dfrac{dy}{dx} = \dfrac{\log x}{(1 + \log x)^2}$.

19. Without expanding, prove that $\begin{vmatrix} a + bx & c + dx & p + qx \\ ax + b & cx + d & px + q \\ u & v & w \end{vmatrix} = \left(1 - x^2\right) \begin{vmatrix} a & c & p \\ b & d & q \\ u & v & w \end{vmatrix}$.

OR

If $A_x = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}$, then prove that $(A_x)^n = \begin{bmatrix} \cos nx & \sin nx \\ -\sin nx & \cos nx \end{bmatrix}$, for all $n \in \mathbb{N}$.

20. Evaluate: $\displaystyle\int_0^{\pi/4} \left(\sqrt{\tan x} + \sqrt{\cot x}\right) dx.$

21. Find the particular solution of the differential equation $\dfrac{dy}{dx} + 2y\ \tan x = \sin x$, given that $y = 0$ when $x = \dfrac{\pi}{3}$.

22. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three mutually perpendicular vectors of equal magnitude, prove that the vector $\vec{a} + \vec{b} + \vec{c}$ is equally inclined to $\vec{a}$, $\vec{b}$ and $\vec{c}$.

OR

Using vectors, find the area of the triangle ABC with vertices $A(1, 2, 3)$, $B(2, -1, 4)$ and $C(4, 5, -1)$.

23. A factory produces bulbs. The probability that any one bulb is defective is $\dfrac{1}{50}$ and they are packed in boxes of 10. From a single box, find the probability that

(i) none of the bulbs is defective.

(ii) more than 8 bulbs work properly.

The manufacturer knows that the product is defective, even then he sells it in the market. Is he doing the right thing? Which life skill is he lacking?

Section D

Question numbers **24** *to* **29** *carry* **6** *marks each.*

24. Let L be the set of all lines in the XY-plane and R be the relation on L defined as $R = \{(L_1, L_2) : L_1 \text{ is parallel to } L_2\}$. Show that R is an equivalence relation. Find the set of all lines related to the line $y = 2x + 4$.

OR

Solve: $\cos\left[\sin^{-1}\dfrac{2}{5} + \cos^{-1}x\right] = 0$.

25. A factory manufactures two types of screws, A and B. Each type of screw requires the use of two machines—an automatic and a hand-operated. It takes 4 minutes on the automatic and 6 minutes on the hand-operated machine to manufacture a package of screws A, while it takes 6 minutes on the automatic and 3 minutes on the hand-operated machine to manufacture a package of screws B. Each machine is available for at most 4 hours on any day. The manufacturer can sell a package of screws A at a profit of ₹ 7 and screws B at a profit of ₹ 10. Assuming that he can sell all the screws he can manufacture, how many packages of each type should he produce in a day in order to maximise his profit? Determine the maximum profit. Express it as an L.P.P. and then solve it.

26. Show that the semi-vertical angle of a right circular cone of a given surface area and maximum volume is $\sin^{-1}\dfrac{1}{3}$.

OR

If the lengths of three sides of a trapezium other than base are equal to 10 cm, then find the area of trapezium when it is maximum.

27. Two schools P and Q want to award their selected students on the values of Discipline, Politeness and Punctuality. The school P wants to award ₹ x each, ₹ y each and ₹ z each for the three respective values to its 3, 2 and 1 students with a total award money of ₹ 1000. School Q wants to spend ₹ 1500 to award its 4, 1 and 3 students on the respective values (by giving the same award money for the three values as before). If the total amount of awards for one prize on each value is ₹ 600, using matrices, find the award money for each value.

28. The area between $x = y^2$ and $x = 4$, which is divided into two equal parts by the line $x = a$. Find the value of a.

29. Evaluate: $\displaystyle\int_0^{\pi} x \;\log\sin x \, dx$

OR

Evaluate: $\displaystyle\int_{-\pi/4}^{\pi/4} \log\left(\sin x + \cos x\right) dx$

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Answers to Unsolved Practice Papers

Answers to Unsolved Practice Paper – 1

1. 0.

2. $(g \circ f)(x) = 2x$, $(f \circ g)(x) = 8x$.

3. $\frac{5\pi}{6}$.

4. 1×18, 18×1, 2×9, 9×2, 3×6 and 6×3; 1×5 and 5×1.

5. $\frac{1}{4}$.

6. $\frac{e^x}{x+1} + C$.

9. 0.784.

10. $(0, 0)$, $(\pi, -2)$ and $(2\pi, 0)$.

12. $\log|\sec x + \tan x| + \log|\sec x| + C$.

13. Mean $= \frac{2}{13}$, Variance $= \frac{400}{2873}$, S.D. $= \frac{20}{13\sqrt{17}}$ **OR** $\frac{2}{3}$.

15. $\cos\left(\frac{y}{x}\right) = \log|x|$.

16. $\frac{1}{44}\begin{bmatrix} 2 & -3 \\ 16 & -2 \end{bmatrix}$.

17. $\frac{2}{75}$ m/s.

19. $\frac{x}{2} + \log|x| - \frac{3}{4}\log|2x - 1| + C$.

21. (i) $\frac{1}{15}$, (ii) $\frac{8}{15}$, (iii) $\frac{2}{5}$, (iv) $\frac{7}{15}$

A person appearing for an interview should be intelligent and honest.

22. $\frac{\pi}{12}$ **OR** $\frac{3}{\pi} + \frac{1}{\pi^2}$.

23. $a = 2$, $b = 1$ **OR** Differentiable.

24. Skew lines, $\frac{6}{\sqrt{5}}$ units **OR** $8x - 4y - z = 0$ and $5x + 2y + 5z - 9 = 0$.

25. $\{(1, 4), (2, 5), (3, 6), (4, 7), (5, 8), (6, 9)\}$ **OR** 13.

26. $\left(abe\sqrt{1 - e^2} + ab\ \sin^{-1} e\right)$ sq. units.

27. Items $P = 2$, Items $Q = 4$ and maximum profit = ₹ 280.

29. $\left(\frac{17}{3}, 0, \frac{23}{3}\right)$.

Answers to Unsolved Practice Paper – 2

2. No

3. $x = \pm 6$

4. $60°$

5. (I) f is strictly increasing on $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ and strictly decreasing on $\left(0, \frac{\pi}{4}\right)$.

(II) f is increasing on $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$ and decreasing on $\left[0, \frac{\pi}{4}\right]$.

6. $2(9!)(10!)(11!)$

7. 4

9. $a = -5$ and $b = 8$

11. -1

12. Yes

13. $-\frac{1}{3}\left(1+\frac{1}{x^2}\right)^{3/2}\log\left(1+\frac{1}{x^2}\right)+\frac{2}{9}\left(1+\frac{1}{x^2}\right)^{3/2}+C$

14. The total requirement of proteins and carbohydrates for family F_1 is 730 g and 5600 g respectively and the total requirement of proteins and carbohydrates for family F_2 is 480 g and 3900 g respectively.

Vitamins, fats and minerals should be included in the diet for a healthy living.

15. Not continuous **OR** $3a = 3b + 2$

16. (i) $\frac{3}{5}$ (ii) $\frac{7}{15}$ **OR** $\frac{13}{30}$

17. $-2x - 3y + 3z + 5 = 0$

18. $\frac{27}{2}$

19. $y\sin x = 2x^2 - \frac{\pi^2}{2}$

20. $\pm\frac{1}{3}(2\hat{i} - 2\hat{j} - \hat{k})$

23. $2\sqrt{2}\,x - 3y = 2$

24. Number of bags of brand P fertiliser $= 40$, number of bags of brand Q fertiliser $= 100$ and minimum amount of nitrogen added in the garden $= 470$ kg.

25. $(5, 2, 6)$, $\sqrt{11}$ units **OR** $(-1, -6, -12)$

26.

X	0	1	2
$P(X)$	$\frac{38}{245}$	$\frac{120}{245}$	$\frac{87}{245}$

, Mean $= \frac{6}{5}$ **OR** $\frac{1}{29}$

27. Not reflexive, Symmetric, Transitive.

29. 27 sq. units.

Answers to Unsolved Practice Paper – 3

1. f^{-1} does not exist.

2. $-\frac{5}{2}$

3. $\frac{5}{12}$

5. -4

6. Not reflexive, Not symmetric, Not transitive

7. $-\frac{1}{2\sqrt{x}\,(1+x)}$

8. $2\left[\frac{(\sqrt{x}+1)^3}{3} - \frac{3(\sqrt{x}+1)^2}{2} + 3(\sqrt{x}+1) - \log|(\sqrt{x}+1)|\right] + C$

9. $\sec(b-a)\,[\log|\sin(x-a)| + \log|\sec(x-b)|] + C$

11. (I) $f(x)$ is strictly increasing on $(0, 1)$ and strictly decreasing on $(1, \infty)$.
(II) $f(x)$ is increasing on $(0, 1]$ and decreasing on $[1, \infty)$.

12. $\left(\frac{1}{4}, \frac{1}{2}\right)$

13. $\frac{\pi^2}{16}$

14. $\log|y| + \frac{1}{y} = -\frac{1}{x} + x + 1$

15. $\frac{1}{\sqrt{1-x^2}} - \frac{2x}{\sqrt{1-x^4}}$

17. $\vec{\beta}_1 = \frac{1}{2}(3\hat{i} - \hat{j}), \vec{\beta}_2 = \frac{1}{2}\,\hat{i} + \frac{3}{2}\,\hat{j} - 3\hat{k}$

18. $\frac{5}{6}\log|1 + 2x + 3x^2| - \frac{11}{3\sqrt{2}}\,\tan^{-1}\left[\frac{3x+1}{\sqrt{2}}\right] + C$

19. $a = 3, b = 2$

21. $\frac{56}{83}$ **OR** 42%

22. $\frac{1}{\pi}$ cm/s. **The values / life skills reflected in this question are bravery, awareness and empathy for children.**

24.

X	0	1	2	3	4
$P(X)$	$\frac{16}{81}$	$\frac{32}{81}$	$\frac{24}{81}$	$\frac{8}{81}$	$\frac{1}{81}$

, Mean $= \frac{4}{3}$ **OR** $\frac{10}{121}$

25. 9

26. 1, 2, 3

27. $(1, 6, 0)$, $\sqrt{24}$ units, $(-3, 8, -2)$

28. $\{1, 5, 9\}$ **OR** $\frac{\sqrt{21}}{14}$

29. Number of working days of $A = 5$, number of working days of $B = 3$ and minimum labour cost = ₹ 1350.

Answers to Unsolved Practice Paper – 4

1. $\frac{5\pi}{6}$

2. 27

3. $45°$

4. $\frac{\pi}{6}$

5. 6

6. $\frac{1}{10}$

8. Commutative, Not associative

9. $\frac{\sqrt{62}}{2}$ sq. units

10. $\sqrt{3}\,b\ \text{cm}^2/\text{s}$

11. $\left(\frac{7}{2}, \frac{1}{4}\right)$

12. $\frac{x+2}{3} = \frac{y-4}{-5} = \frac{z+5}{6}$

15. $\frac{20}{21}, \frac{113}{180}$

16. $4x - 3y + 2z = 3$

17. $\frac{1}{\sqrt{3}}\,\tan^{-1}\left(\frac{\tan^2 x - 1}{\sqrt{3}\tan x}\right) + C$

20. $\frac{\pi}{\sqrt{2}}$

21. $y\sec^2 x = \sec x - 2$

22. $\frac{\sqrt{274}}{2}$ sq. units

23. (i) $\left(\frac{49}{50}\right)^{10}$ (ii) $\frac{59}{50}\left(\frac{49}{50}\right)^{9}$. **It is not right to sell defective items in the market. The life skill that the manufacturer lacks is honesty.**

24. $\{y = 2x + c : c \in \mathbb{R}\}$ **OR** $\frac{2}{5}$

25. Number of packages of screw $A = 30$, number of packages of screw $B = 20$ and maximum profit = ₹ 410.

26. $75\sqrt{3}\ \text{cm}^2$

27. The award money for discipline, politeness and punctuality is ₹ 100, ₹ 200 and ₹ 300 respectively.

28. $(4)^{2/3}$

29. $-\frac{\pi^2}{2}\log 2$ **OR** $-\frac{\pi}{4}\log 2$

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Mock Examination Paper
2017

Mock Examination Paper 2017

Time: 3 Hours **Max. Marks: 100**

General Instructions

(i) **All** questions are compulsory.

(ii) Please check that this question paper consists of **29** questions.

(iii) Questions **1** to **4** in Section A are Very Short Answer Type Questions carrying **1** mark each.

(iv) Questions **5** to **12** in Section B are Short Answer I Type Questions carrying **2** marks each.

(v) Questions **13** to **23** in Section C are Long Answer I Type Questions carrying **4** marks each.

(vi) Questions **24** to **29** in Section D are Long Answer II Type Questions carrying **6** marks each.

(vii) Please write down the serial number of the question before attempting it.

Section A

Question numbers **1** *to* **4** *carry* **1** *mark each.*

1. Evaluate: $\int_{-\pi}^{\pi} (\sin^{83} x + x^{123})\, dx$.

2. Find the value of $\hat{i} \cdot (\hat{j} \times \hat{k}) + \hat{j} \cdot (\hat{i} \times \hat{k}) + \hat{k} \cdot (\hat{i} \times \hat{j})$.

3. If A is an invertible matrix of order 3×3 and $|A| = 5$, then find $|\operatorname{adj} A|$.

4. Find the general solution of the differential equation

$$y_1 = \frac{ae^{2x} + ae^{4x}}{e^x + e^{-x}}.$$

Section B

Question numbers **5** *to* **12** *carry* **2** *marks each.*

5. If $\Delta = \begin{vmatrix} 0 & b-a & c-a \\ a-b & 0 & c-b \\ a-c & b-c & 0 \end{vmatrix}$, then show that Δ is equal to zero.

6. A coin is tossed three times. Determine $P(E|F)$, where E : head on third toss and F : head occurs on first two tosses.

7. Determine whether the relation R on $\mathbb{N}$ defined by, $R = \{(x, y) : y = x + 5, x < 4\}$, is reflexive, symmetric and transitive.

8. Evaluate: $\int_0^{\pi/2} \log \tan x \, dx$

9. Differentiate $\cos x^x$ w.r.t. x.

10. If $\vec{a} = \hat{i} - \hat{j} + 7\hat{k}$ and $\vec{b} = 5\hat{i} - \hat{j} + \lambda\hat{k}$, then find the value of λ so that $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ are perpendicular vectors.

11. Evaluate: $\int \frac{\sqrt{1 + x^2}}{x^4}\, dx$

12. If $f(x) = \frac{4x + 3}{6x - 4}, x \neq \frac{2}{3}$, then show that $(f \circ f)(x) = x$, for all $x \neq \frac{2}{3}$. What is the inverse of f?

Section C

Question numbers **13** *to* **23** *carry* **4** *marks each.*

13. Prove, using properties of determinants: $\begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix} = a^3+b^3+c^3-3abc.$

OR

If X and Y are 2×2 matrices, then solve the following matrix equations for X and Y:

$$2X+3Y=\begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}, 3X+2Y=\begin{bmatrix} -2 & 2 \\ 1 & -5 \end{bmatrix}$$

14. A die is thrown three times. Let X be 'the number of 2's seen'. Find the expectation of X.

15. Find a vector of magnitude 5 units and parallel to the resultant of the vectors $\vec{a}=2\hat{i}+3\hat{j}-\hat{k}$ and $\vec{b}=\hat{i}-2\hat{j}+\hat{k}$.

OR

Find a vector of magnitude 11 in the direction opposite to that of $\overrightarrow{PQ}$, where P and Q are the points $(1,3,2)$ and $(-1,0,8)$ respectively.

16. If $\sin y = x\sin(a+y)$, then prove that $\dfrac{dy}{dx}=\dfrac{\sin^2(a+y)}{\sin a}$.

17. Evaluate: $\displaystyle\int_0^{\pi/2} \sin 2x\ \tan^{-1}(\sin x)\, dx$

18. Find the equation of a plane which is at a distance of $3\sqrt{3}$ units from origin and the normal to which is equally inclined to the coordinate axes.

19. If $(\cos x)^y=(\cos y)^x$, then find $\dfrac{dy}{dx}$.

OR

Find the approximate value of $f(3.02)$, up to 2 places of decimal, where $f(x)=3x^2+5x+3$.

20. A driver starts a car from a point P at time $t=0$ seconds and stops at point Q. The distance x (in metres) covered by it in t seconds is given by $x=t^2\left(2-\dfrac{t}{3}\right)$. Find the time taken by it to reach Q and also find the distance between P and Q. **The driver has stopped the car at the point Q on the roadside to take the call on his mobile phone. Has he done right in doing so?**

21. Find the particular solution of the differential equation $\dfrac{dy}{dx}=\dfrac{x(2\log x+1)}{\sin y+y\ \cos y}$, given that $y=\dfrac{\pi}{2}$ when $x=1$.

22. Suppose 5 men out of 100 and 25 women out of 1000 are orators. An orator is chosen at random. Find the probability of a male person being selected, assuming that there are equal number of men and women.

23. Evaluate: $\displaystyle\int (x-3)\sqrt{x^2+3x-18}\, dx$

Section D

Question numbers **24** *to* **29** *carry* **6** *marks each.*

24. A diet for a sick person must contain at least 4000 units of vitamins, 50 units of proteins and 1400 calories. Two foods A and B are available at a cost of ₹ 40 and ₹ 30 per unit respectively. If one unit of A contains 200 units of vitamins, 1 unit of proteins and 40 calories, and one unit of food B contains 100 units of vitamins, 2 units of proteins and 40 calories, find what combination of foods should be used to have the least cost. Formulate the above L.P.P. mathematically and then solve it graphically.

25. Find the area of the region $\{(x, y) : y^2 \leq 4x;\ \ 4x^2 + 4y^2 \leq 9\}$.

OR

Show that the normal at any point θ to the curve $x = a\cos\theta + a\theta\sin\theta$, $y = a\sin\theta - a\theta\cos\theta$ is at a constant distance from the origin.

26. Show that the function $f : \mathbb{N} \to \mathbb{N}$, given by $f(x) = x - (-1)^x$, is a bijection.

OR

Let $A = \mathbb{N} \times \mathbb{N}$ and $*$ be a binary operation on A, defined by $(a, b) * (c, d) = (a + c, b + d)$. Show that $*$ is commutative and associative. Also, find the identity element for $*$ on A, if any.

27. Find the equation of the plane that contains the lines

$$\vec{r} = (\hat{i} + \hat{j}) + \lambda(\hat{i} - \hat{j} + 2\hat{k}) \text{ and } \vec{r} = (\hat{i} + \hat{j}) + \mu(\hat{i} + 2\hat{j} - \hat{k}).$$

28. Find the inverse of $\begin{bmatrix} 0 & 2 & -1 \\ 0 & 3 & 1 \\ 3 & 2 & 4 \end{bmatrix}$, using Elementary Row Transformation method.

29. The combined resistance R of two resistors R_1 and R_2 (R_1, $R_2 > 0$) is given by $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2}$. If $R_1 + R_2 = C$ (a constant), show that the maximum resistance R is obtained by choosing $R_1 = R_2$.

OR

A rectangle is inscribed in a semicircle of radius R with one of its sides on the diameter of the semicircle. Find the dimensions of the rectangle so that its area is maximum.

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